Lecture 29

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When $h = 0$, a phase transition occurs at $\beta = 1$. We saw that in the high temperature phase, $R_{12} = O(N^{-1/2})$. Parisi conjectures that at $\beta = 1$ and $h = 0$, R_{12} is of order $N^{-1/3}$. Guerra proved that at $\beta = 1$ and $h = 0$, $\mathbf{E}\langle R_{12}^2\rangle \le C/\sqrt{N}$ for some constant C, ie, $R_{12} = O(N^{-1/4})$ at most. Talagrand has another proof in his book, but it is complicated. Nothing better is known.

Using Stein's method, we will show that $\mathbf{E}\langle |R_{12}|^3\rangle \ge C/N$ for some $C > 0$.

Proof: Let $\psi(x)$ be the probability density

$$\frac{\cosh(x)e^{-x^2/2}}{\sqrt{2\pi e}}$$

(i.e. the symmetric mixture of $N(1, 1)$ and $N(-1, 1)$). Hopefully, this is the distribution of the local field as $N \to \infty$.

For any bounded, measurable f, let $Mf = \int_{-\infty}^{\infty} f(x)\psi(x)\,dx$. Define an operator U as

$$Uf(x) = \frac{e^{x^2/2}}{\cosh x}\int_{-\infty}^{\infty} \cosh(t)e^{-t^2/2}(f(t) - Mf)\,dt$$

and let $Tf(x) = f'(x) - (x - \tanh(x))f(x)$. Verify that $TUf = f - Mf$, so U is the inversion of the Stein operator.

Lemma 1 *$||Uf||_\infty \le C||f||_\infty$ and $||(Uf)'||_\infty \le C||f||_\infty$ for some universal constant C.*

Lemma 2 (Expansion lemma) *Fix a bounded, measurable $f : \mathbb{R} \to \mathbb{R}$, let $b_1, \ldots, b_m$ be arbitrary functions of σ. Assume that b_1 does not depend on σ_1. Then*

$$\begin{aligned}\mathbf{E}(\langle f(l_1)b_1\rangle\langle b_2\rangle\cdots\langle b_m\rangle) =& (Mf)\mathbf{E}(\langle b_1\rangle\cdots\langle b_m\rangle)\\ &- \sum_{r=2}^{m}\frac{1}{N}\sum_{j=2}^{N}\mathbf{E}(\langle\sigma_j Uf(l_1)b_1\rangle\langle b_2\rangle\cdots\langle b_r\sigma_1\sigma_j\rangle\langle b_{r+1}\rangle\cdots\langle b_m\rangle)\\ &+ \frac{m}{N}\sum_{j=2}^{N}\mathbf{E}(\langle\sigma_j Uf(l_1)b_1\rangle\langle b_2\rangle\cdots\langle b_m\rangle) + \frac{1}{N}\mathbf{E}(\langle(Uf)'(l_1)b_1\rangle\langle b_2\rangle\cdots\langle b_m\rangle)\\ &+ \frac{1}{N}\mathbf{E}(\langle\sigma_j Uf(l_1)b_1\sigma_1\rangle\langle b_2\rangle\cdots\langle b_m\rangle)\end{aligned}$$

Now take $f(x) = \tanh(x)$. Then $\langle\sigma_1\sigma_2\rangle = \langle f(l_1)\sigma_2\rangle$ (since $f(l_1)$ is the conditional expectation of σ_1 given the rest of the spins). Thus $\mathbf{E}\langle\sigma_1\sigma_2\rangle^2 = \mathbf{E}(\langle f(l_1)\sigma_2\rangle\langle\sigma_1\sigma_2\rangle)$.

Let $b_1 = \sigma_2$, $b_2 = \sigma_1\sigma_2$. Since ψ is a symmetric density and tanh is odd, $Mf = 0$. Let $h = Uf$. Applying the expansion lemma,

$$\begin{aligned}\mathbf{E}\langle\sigma_1\sigma_2\rangle^2 =& 0 - \frac{1}{N}\sum_{j=2}^{N}\mathbf{E}(\langle\sigma_1 h(l_1)\sigma_2\rangle\langle\sigma_2\sigma_j\rangle)\\ &+ \frac{2}{N}\sum_{j=2}^{N}\mathbf{E}(\langle\sigma_j h(l_1)\sigma_2\rangle\langle\sigma_2\sigma_j\rangle\langle\sigma_1\sigma_j\rangle)\\ &+ \frac{1}{N}\mathbf{E}(\langle h'(l_1)\sigma_2\rangle\langle\sigma_1\sigma_2\rangle)\\ &+ \frac{1}{N}\mathbf{E}(\langle h(l_1)\sigma_1\sigma_2\rangle\langle\sigma_1\sigma_2\rangle)\end{aligned}$$

To simplify the above, let *Rem* denote any term that is bounded by $CN^{-1}\sqrt{\mathbf{E}\langle R_{12}\rangle^2}$ for some constant C.

Since h' is bounded, so

$$\begin{aligned}\frac{1}{N}\mathbf{E}(\langle h'(l_1)\sigma_2\rangle\langle\sigma_1\sigma_2\rangle) &\le \frac{C}{N}\mathbf{E}|\langle\sigma_1\sigma_2\rangle| \le \frac{C}{N}\sqrt{\mathbf{E}\langle\sigma_1\sigma_2\rangle^2}\\ &\le \frac{C'}{N}\sqrt{\mathbf{E}\frac{1}{N^2}\sum_{i,j}\langle\sigma_i\sigma_j\rangle^2} = \frac{C'}{N}\sqrt{\mathbf{E}\langle R_{12}\rangle^2}.\end{aligned}$$

So the fourth term is *Rem*. The last term is also *Rem*. Any single term in the sum in the third term is also *Rem*. In the 2nd term, for $j = 2$, we get $\frac{1}{N}\mathbf{E}\langle h(l_1)\rangle$. All other terms are *Rem*. So

$$E\langle\sigma_1\sigma_2\rangle^2 = -\frac{1}{N}\mathbf{E}\langle h(l_1)\rangle - \mathbf{E}(\langle h(l_1)\sigma_2\sigma_3\rangle\langle\sigma_2\sigma_3\rangle) - 2\mathbf{E}(\langle h(l_1)\sigma_2\sigma_3\rangle\langle\sigma_1\sigma_2\rangle\langle\sigma_1\sigma_3\rangle) - Rem.$$

This is the first expansion. To complete the proof, we apply the expansion lemma to each of the above terms. It will be enough to show:

$$-\frac{\mathbf{E}\langle h(l_1)\rangle}{N} = \frac{1}{N} + Rem,$$

$$-\mathbf{E}(\langle h(l_1)\sigma_2\sigma_3\rangle\langle\sigma_2\sigma_3\rangle) = E\langle\sigma_1\sigma_2\rangle^2 + T_1,$$

where $|T_1| \le C\mathbf{E}\langle|R_{12}|^3\rangle$, and

$$|\mathbf{E}(\langle h(l_1)\sigma_2\sigma_3\rangle\langle\sigma_1\sigma_2\rangle\langle\sigma_1\sigma_3\rangle)| \le C\mathbf{E}\langle|R_{12}|^3\rangle + Rem.$$

If we can show the above, then

$$\mathbf{E}\langle\sigma_1\sigma_2\rangle^2 = \frac{1}{N} + \mathbf{E}\langle\sigma_1\sigma_2\rangle^2 + T_1 + T_2$$

where T_1, T_2 are bounded by $C\mathbf{E}\langle|R_{12}|^3\rangle + Rem$. So

$$\frac{1}{N} \le C\mathbf{E}\langle|R_{12}|^3\rangle + Rem \le C\mathbf{E}\langle|R_{12}|^3\rangle + \frac{C'\sqrt{\mathbf{E}\langle R_{12}^2\rangle}}{N}$$

Now $\sqrt{\mathbf{E}\langle R_{12}^2\rangle} \le (\mathbf{E}\langle|R_{12}|^3\rangle)^{1/3}$. Suppose $\mathbf{E}\langle|R_{12}|^3\rangle \le \frac{1}{2CN}$. Then we get $1/N \le 1/2N + (C'/N)(1/2CN)^{1/3}$, a contradiction when N is large enough. So for N large enough, $\mathbf{E}\langle|R_{12}|^3\rangle \ge \frac{1}{2CN}$.

To prove the above three statements:

Apply the approximation lemma, and only the first terms will matter. Let $w = Uh$ (so we invert the Stein operator again). Verify that $Mh = -1$. By the expansion lemma,

$$\begin{aligned}
\mathbf{E}\langle h(l_1)\rangle &= -1 + \frac{1}{N}\sum_{j=2}^{N}\mathbf{E}(\langle\sigma_j w(l_1)\rangle\langle\sigma_1\sigma_j\rangle) + \frac{\mathbf{E}\langle w'(l_1)\rangle + \mathbf{E}\langle\sigma_1\sigma_j\rangle}{N} \\
&= -1 + Rem + O(1/N).
\end{aligned}$$

Using the expansion lemma on the second term, $\mathbf{E}(\langle h(l_1)\sigma_2\sigma_3\rangle\langle\sigma_2\sigma_3\rangle) = (Mh)\mathbf{E}\langle\sigma_2\sigma_3\rangle^2 = -\mathbf{E}\langle\sigma_2\sigma_3\rangle^2 = -\mathbf{E}\langle\sigma_1\sigma_2\rangle^2$ with some remainder terms. The third term can also be bounded through the expansion lemma.

Lastly, a sketch of the proof of the expansion lemma:

We want to find $\mathbf{E}\langle f(l_1)b_1\rangle\cdots\langle b_m\rangle$. Let $h = Uf$ so that $h'(x) - (x - \tanh(x))h(x) = f(x) - Mf$. Replace $f(l_1) - Mf$ by $h'(l_1) - l_1 h(l_1) + \tanh(l_1)h(l_1)$ and apply integration by parts on the terms arising from $l_1 h(l_1)$.

□

Exercise 3 *Get an upper bound for* $\mathbf{E}\langle|R_{12}|^3\rangle$.

Exercise 4 *Evaluate*

$$\lim_{N\to\infty}\mathbf{E}(N\langle R_{12}R_{23}R_{31}\rangle)$$

or, alternatively,

$$\lim_{N\to\infty}\mathbf{E}(N\langle\sigma_1\sigma_2\rangle\langle\sigma_2\sigma_3\rangle\langle\sigma_3\sigma_1\rangle)$$

You can use Guerra's result.