Lecture 28

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1 A recap

For $\beta < 1/2$ in the Sherrington-Kirkpatric (S-K) model, we showed a bound on the overlap R_{12}:

$$\mathbf{E}\left\langle (R_{12} - q)^{2k} \right\rangle \leq \frac{(Ck)^k}{N^k} \ , \tag{1}$$

where $q = \mathbf{E}[\tanh^2(\beta Z \sqrt{q} + h)]$ and $Z \sim \mathcal{N}(0, 1)$. This means that the overlap is concentrated. When $h = 0$ this implies that

$$\mathbf{E}\left\langle R_{12}^{2k} \right\rangle \leq \frac{(Ck)^k}{N^k} \ , \tag{2}$$

so R_{12} is close to 0 in this case. This result was crucial in showing that for $h = 0$ the quantity

$$l_1 = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \sigma_j \tag{3}$$

has a limiting annealed distribution

$$\frac{1}{2}\mathcal{N}(\beta, 1) + \frac{1}{2}\mathcal{N}(-\beta, 1) \ . \tag{4}$$

We also proved that the quenched distribution of l_1 converges in probability to (4) by showing that

$$\left\langle f'(l_1) - (l_1 - \beta \tanh(\beta l_1)) f(l_1) \right\rangle \xrightarrow{P} 0 \ . \tag{5}$$

Note that this convergence is in probability on the space of probability measures.

Finally, we also found a CLT for the Hamiltonian $\sum g_{ij} \sigma_i \sigma_j$ when $h = 0$. When $h \neq 0$ the limiting distribution of the Hamiltonian is not known.

2 The TAP equations

Today we will start looking at the Thouless-Anderson-Palmer (TAP) equations, which are a collection of self-consistent equations for the quenched average value for $i = 1, 2, \ldots, N$:

$$\langle \sigma_i \rangle \approx \tanh\left(\frac{\beta}{\sqrt{N}} \sum_{j=1, j\neq i}^{N} g_{ij} \langle \sigma_j \rangle + h - \beta^2 (1-q) \langle \sigma_i \rangle \right) \tag{6}$$

Furthermore, it is true that

$$\langle \sigma_i \rangle \xrightarrow{d} \tanh(\beta z \sqrt{q} + h) \tag{7}$$

where $z \sim \mathcal{N}(0,1)$. Moreover, $\langle \sigma_1 \rangle, \langle \sigma_2 \rangle, \ldots, \langle \sigma_p \rangle$ are asymptotically independent for fixed p as $N \to \infty$.

The concentration of the overlaps implies that

$$\langle \sigma_1 \sigma_j \rangle \cong \langle \sigma_i \rangle \langle \sigma_j \rangle \ , \tag{8}$$

which in turn implies

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \sigma_i &\cong \left\langle \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right\rangle && (9) \\ &= \frac{1}{N} \langle \sigma_i \rangle && (10) \\ &\xrightarrow{P} \mathbf{E}[\tanh(\beta z \sqrt{q} + h)] \ . && (11) \end{aligned}$$

Similarly, since $R_{12} \to q$,

$$\begin{aligned} R_{12} &\cong \langle R_{12} \rangle && (12) \\ &= \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle^2 && (13) \\ &\xrightarrow{P} \mathbf{E}[\tanh^2(\beta z \sqrt{q} + h)] \ . && (14) \end{aligned}$$

This shows why q must satisfy

$$q = \mathbf{E}[\tanh^2(\beta z \sqrt{q} + h)] \ . \tag{15}$$

For simplicity of notation, let us define

$$r_i = \frac{1}{\sqrt{N}} \sum_{j=1, j\neq i}^{N} g_{ij} \langle \sigma_j \rangle - \beta(1-q) \langle \sigma_i \rangle \tag{16}$$

so that the TAP equations say

$$\langle \sigma_i \rangle \cong \tanh(\beta r_i + h) \ . \tag{17}$$

Note that r_i is a function of g only. It can be shown that $r_i \xrightarrow{d} \mathcal{N}(0,q)$.

3 A sketch of the proof

First, note that the conditional expectation of σ_1 given $\sigma_2, \ldots, \sigma_N$ is just $\tanh(\beta l_1 + h)$, so

$$\langle \sigma_1 \rangle = \langle \tanh(\beta l_1 + h) \rangle \ . \tag{18}$$

Now the goal is to approximate the distribution of l_1.

We first reparameterize Gaussian mixtures. Given a, b, μ, σ^2, let ψ_{a,b,μ,σ^2} denote the probability density on $\mathbb{R}$ proportional to

$$\cosh(ax + b) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \ . \tag{19}$$

Exercise Show that ψ_{a,b,μ,σ^2} is the same as $p\varphi_{\mu_1,\sigma^2} + (1-p)\varphi_{\mu_2,\sigma^2}$, where φ_{μ,σ^2} is the density $\mathcal{N}(\mu, \sigma^2)$, $\mu_1 = \mu + a\sigma^2$, $\mu_2 = \mu - a\sigma^2$, and

$$p = \frac{\exp(a\mu + b)}{\exp(a\mu + b) + \exp(-a\mu - b)} \ . \tag{20}$$

If $X \sim \psi_{a,b,\mu,\sigma^2}$, then

$$\mathbf{E}[\tanh(aX + b)] = \tanh(a\mathbf{E}[X] + b - (2p-1)a^2\sigma^2) \tag{21}$$

$$= \tanh(a\mu + b) \ . \tag{22}$$

The term $-(2p-1)a^2\sigma^2$ is called the Onsager correction term, and is what allows us to move the expectation inside the tanh. The quenched distribution of l_1 is approximately $\psi_{\beta,h,r_1,1-q}$, and so

$$\langle \tanh(\beta l_1 + h) \rangle \cong \tanh(\beta r_1 + h) \tag{23}$$

and $r_1 = \langle l_1 \rangle - \beta(1-q)\langle \sigma_1 \rangle$. The quenched distribution is a random distribution with parameter r_1.

The Stein characterizing operator for ψ_{a,b,μ,σ^2} is

$$Tf(x) = f'(x) - \left(\frac{x-\mu}{\sigma^2} - a\tanh(ax+b)\right) f(x) \tag{24}$$

To see this, look at

$$f'(x) + \left(\frac{d}{dx} \log \psi_{a,b,\mu,\sigma^2}(x)\right) f(x) \ . \tag{25}$$

Recall that for the characteristic operator, if $X \sim \psi_{a,b,\mu,\sigma^2}$ then $\mathbf{E}[Tf(x)] = 0$ for all f and conversely.

We have to show that

$$\mathbf{E}\left\langle f'(l_1) - \left(\frac{l_1 - r_1}{1-q} - \beta\tanh(\beta l_1 + h)\right) f(l_1)\right\rangle^2 \longrightarrow 0 \ . \tag{26}$$

It is instructive to consider the contrast with the annealed equation. If $\mathbf{E}\langle\cdot\rangle \to 0$ then we've proved nothing. This comes from r_1 not being a constant. However, a quenched equation implies a distributional result because r_1 is a constant, given g.

The remaining steps are then

1. Start with

$$h_j(g) = \frac{1}{\sqrt{N}}\langle(\sigma_j - \langle\sigma_j\rangle)f(l_1)\rangle \ . \tag{27}$$

2. Then use the approximation Lemma to show that

$$\sum_{j=2}^{N} g_{1j}h_j \cong \sum_{j=2}^{N}\frac{\partial h_j}{\partial g_{1j}} \ . \tag{28}$$

3. Recognize, after some computation and using $R_{12} \cong q$, that (26) and (28) are the same.

The full details of these arguments can be found in the paper (S. Chatterjee, *Spin Glasses and Stein's Method*, arXiv:0706.3500v1 [math.PR]).