Lecture 26

Lecture date: October 26, 2007 *Scribe: Richard Liang*

1 Annealed CLT for the Hamiltonian of the Sherrington-Kirkpatrick model

We begin by recalling the setting of the previous lecture: let $\beta < 1/2$, $h = 0$, and

$$l_i = \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} g_{ij} \sigma_j.$$

We saw that the annealed distribution of l_1 approaches

$$\frac{1}{2} N(\beta, 1) + \frac{1}{2} N(-\beta, 1)$$

as $N \to \infty$. (In fact this can be extended to the regime $\beta < 1$, $h = 0$; we'll return to this later.) We also saw that for any smooth function f,

$$\nu\big(f'(l_1) - (l_1 - \beta \tanh(\beta l_1))\, f(l_1)\big) \to 0$$

(here ν is the annealed distribution).

Exercise 1 *Develop Stein's method for*

$$\frac{1}{2} N(\beta, 1) + \frac{1}{2} N(-\beta, 1)$$

and get a total variation bound for the above CLT.

The next exercise should have been given earlier, but is an important result.

Exercise 2 *Recall Lemma 3.4 from the proof of the KMT embedding: if W is a random variable such that $\mathbf{E}W = 0$ and $\mathbf{E}\big[W^2\big] < \infty$, and T is a random variable such that*

$$\mathbf{E}[W \varphi(W)] = \mathbf{E}\big[T \varphi'(W)\big] \text{ for all } \varphi \text{ Lipschitz,} \tag{1}$$

then for all $\sigma^2 > 0$, there exists a pair of random variables (W', Z) on the same probability space such that W' is a version of W, $Z \sim N(0, \sigma^2)$ and

$$\mathbf{E}\big[\exp(\theta\,|W' - Z|)\big] \le 2\mathbf{E}\left[\exp\left(\frac{2\theta^2\left(T - \sigma^2\right)^2}{\sigma^2}\right)\right].$$

Prove a multivariate version of this result.

((1) can be alternately written as $\mathbf{E}[W\varphi'(W)] = \mathbf{E}[T\varphi''(W)]$ for all φ. This generalizes to $\mathbf{E}[W \cdot \nabla\varphi(W)] = \mathbf{E}[Tr(T\, Hess(\varphi(W)))]$, where $W = (W_1, \ldots, W_d)$, $\varphi : \mathbb{R}^d \to \mathbb{R}$, and $T = (T_{ij})_{i,j \le d}$ is a positive definite random matrix. Replace σ^2 by the positive definite matrix $\Sigma = (\sigma_{ij})_{i,j=1}^d$ and replace θ by $\theta \in \mathbb{R}^d$.)

In the regime $\beta < 1/2$, $h = 0$, we'll compute the limiting annealed distribution of the Hamiltonian

$$\sum_{i<j} g_{ij}\sigma_i\sigma_j.$$

Specifically, we will consider

$$H = \frac{1}{N}\sum_{i<j} g_{ij}\sigma_i\sigma_j - \frac{\sqrt{N}\beta}{2}.$$

The appropriate scaling and centering come from the following ("very simple") exercise.

Exercise 3 *Let*

$$p_N(\beta) = \mathbf{E}\left[\frac{\log Z_N(\beta)}{N}\right].$$

We saw that $p_N(\beta) \to \log 2 + \beta^2/4$. Show

$$p_N'(\beta) = \mathbf{E}\left\langle \frac{1}{N^{3/2}}\sum_{i<j} g_{ij}\sigma_i\sigma_j \right\rangle$$

$$p_N''(\beta) = \mathbf{E}\mathrm{Var}\left(\frac{1}{N}\sum_{i<j} g_{ij}\sigma_i\sigma_j \,\middle|\, g\right).$$

Take any smooth f: we have

$$\begin{aligned}
\nu(Hf(H)) &= \mathbf{E}\left\langle \frac{1}{N}\sum_{i<j} g_{ij}\sigma_i\sigma_j f(H)\right\rangle - \frac{\sqrt{N}\beta}{2}\mathbf{E}\langle f(H)\rangle \\
&= \frac{1}{N}\sum_{1\le i<j\le N} \mathbf{E}[g_{ij}\,\langle\sigma_i\sigma_j f(H)\rangle] - \frac{\sqrt{N}\beta}{2}\mathbf{E}\langle f(H)\rangle. \qquad (2)
\end{aligned}$$

We use integration by parts to deal with the first term in (2):

$$\begin{aligned}
&\frac{1}{N}\sum_{1\le i<j\le N}\mathbf{E}[g_{ij}\langle\sigma_i\sigma_j f(H)\rangle] \\
&= \frac{1}{N}\sum_{i<j}\mathbf{E}\left[\frac{\partial}{\partial g_{ij}}\langle\sigma_i\sigma_j f(H)\rangle\right] \\
&= \frac{1}{N^2}\sum_{i<j}\mathbf{E}\left\langle f'(H)\right\rangle - \frac{\beta}{N^{3/2}}\sum_{i<j}\mathbf{E}[\langle f(H)\sigma_i\sigma_j\rangle\langle\sigma_i\sigma_j\rangle] + \frac{\beta}{N^{3/2}}\sum_{i<j}\mathbf{E}\langle f(H)\rangle. \qquad (3)
\end{aligned}$$

The first term in (3) equals

$$\frac{N(N-1)}{2N^2}\nu\big(f'(H)\big) \approx \frac{1}{2}\nu\big(f'(H)\big)$$

and similarly the third is approximately

$$\frac{\sqrt{N}\beta}{2}\nu(f(H)),$$

which takes care of the second term in (2). The second term of (3) is approximately

$$\frac{\beta}{2N^{3/2}}\mathbf{E}\left[\sum_{i,j=1}^{N}\langle f(H)\sigma_i\sigma_j\rangle\langle\sigma_i\sigma_j\rangle\right].$$

We can write the summand as $\left\langle f\big(H\big(g,\sigma^1\big)\big)\,\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right\rangle$, where σ^1 and σ^2 are two spin configurations independently drawn from the quenched distribution $\langle\cdot\rangle$. Doing this, the above gives

$$\begin{aligned}
&\frac{\beta\sqrt{N}}{2}\mathbf{E}\left\langle f\big(H\big(g,\sigma^1\big)\big)\,\frac{1}{N^2}\sum_{i,j=1}^{N}\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right\rangle \\
&= \frac{\beta\sqrt{N}}{2}\mathbf{E}\left\langle f\big(H\big(g,\sigma^1\big)\big)\,R_{12}^2\right\rangle.
\end{aligned}$$

If f is bounded, then

$$\left|\mathbf{E}\left\langle f(H)R_{12}^2\right\rangle\right| \le |f|_\infty\mathbf{E}\left\langle R_{12}^2\right\rangle \le \frac{C|f|_\infty}{N}.$$

Putting these above calculations back into (2), we get

$$\nu(Hf(H)) = \frac{1}{2}\nu\big(f'(H)\big) + O\left(\frac{1}{\sqrt{N}}\right),$$

which shows that $H \Longrightarrow N(0,1/2)$.

Exercise 4 *Prove an annealed CLT for* $\sum_{i<j} g_{ij}\sigma_i\sigma_j$ *when* $\beta < 1/2, h \neq 0$.

2 Quenched laws

Example 5 *For the Hamiltonian, we showed that for any f,*

$$\nu\left(\frac{1}{2}f'(H) - Hf(H)\right) = \mathbf{E}\left\langle \frac{1}{2}f'(H) - Hf(H)\right\rangle \to 0.$$

For a quenched CLT, we have to show that for all f,

$$\left\langle \frac{1}{2}f'(H) - Hf(H)\right\rangle \xrightarrow{p} 0.$$

Exercise 6 *Suppose that for "all" f,*

$$\left\langle \frac{1}{2}f'(H) - Hf(H)\right\rangle \xrightarrow{p} 0.$$

Show that "for all" φ, $\langle \varphi(H)\rangle \xrightarrow{p} \mathbf{E}[\varphi(Z)]$, where $Z \sim N(0, 1/2)$.

Exercise 7 *Suppose Exercise 6 holds. Let μ_N denote the (random) distribution of H under the Gibbs measure (that is, the "quenched" or conditional distribution given g). Show that $\mu_N \xrightarrow{p} N(0, 1/2)$ on the space of probability measures.*