Lecture 25

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Latala's Result: If $\beta < 1/2$ then for $2s < 1 - 4\beta^2$

$$\nu(\exp(sN(R_{1,2} - q)^2)) \leq \frac{1}{\sqrt{1 - 2s - 4\beta^2}} \Rightarrow \nu((R_{1,2} - q)^{2k}) \leq \frac{(Ck)^k}{N^k} \text{ for all } k$$

so the overlap is concentrated.

Exercise 1: Using this result show that for any fixed p,

$$\mathbf{E}(\langle \sigma_1 \sigma_2 \cdots \sigma_p \rangle - \langle \sigma_1 \rangle \langle \sigma_2 \rangle \cdots \langle \sigma_p \rangle)^2 \leq K(p)\nu((R_{1,2} - q)^2) \leq \frac{K(p)}{N}$$

Exercise 2: The above expectation goes to 0 even if p grows with N. How fast can it grow?

The 1st exercise means that any collection of spins at p locations are approximately independent.

Hints for Exercise 1: use induction on p and note if $\sigma^1, \ldots, \sigma^4$ are 4 configurations then $R_{1,3} - R_{1,4} - R_{2,3} + R_{3,4} = \frac{(\sigma^1 - \sigma^2)\cdot(\sigma^3 - \sigma^4)}{N}$.

Exercise 3: (A Talagrand research problem) Show the total variation distribution distance between the joint law of $\sigma_1, \ldots \sigma_p$ and the product of marginals $\to 0$ as $N \to \infty$.

$\frac{d}{dt} \mathbf{E} \frac{\log Z_t}{N} = -\frac{\beta^2}{4}\nu_t((R_{1,2} - q)^2) + \frac{\beta^2}{4}(1 - q)^2$. Since $\nu_t((R_{1,2} - q)^2) = O(\frac{1}{N}) \ \ \forall\ 0 \leq t \leq 1$ thus,

$$\mathbf{E} \frac{\log Z_N}{N} = \varphi(1) = \varphi(0) + \frac{\beta^2(1 - q)^2)}{4} + O(\frac{1}{N}) = \log 2 + \mathbf{E}(\log\cosh(\beta Z\sqrt{q} + h))$$

Thouless - Anderson - Palmer Equations: The random quantities $\langle \sigma_1 \rangle, \langle \sigma_2 \rangle, \ldots \langle \sigma_N \rangle$ satisfy an approximate system of equations

$$\langle \sigma_i \rangle \approx \tanh(\frac{\beta}{\sqrt{N}} \sum_{j=1, j\neq i}^{N} g_{ij}\langle \sigma_j \rangle + h - \beta^2(1 - q)\langle \sigma_i \rangle) \quad i = 1, 2, \ldots, N$$

Talagrand in 2003 gave the first rigorous proof.

Suppose $\beta < 1/2, h = 0$. Then $q = 0$, and so $\mathbf{E}\langle R_{1,2}\rangle \le \frac{c}{N}$. Let $l_i = \frac{1}{\sqrt{N}} \sum_{j=1, j\neq i}^N g_{ij}\sigma_j$ be the "local field" at site i. We will look at the annealed (i.e. unconditional) distribution of l_1.

Take any smooth function $f : \mathbb{R} \to \mathbb{R}$

$$\nu(l_1 f(l_1)) = \frac{1}{\sqrt{N}} \sum_{j=2}^N \mathbf{E}[g_{1j}\langle \sigma_j f(l_1)\rangle] = \frac{1}{\sqrt{N}} \sum_{j=2}^N \mathbf{E}[\frac{\partial}{\partial g_{1j}} \langle \sigma_j f(l_1)\rangle]$$

$$\frac{\partial}{\partial g_{1j}}\langle \sigma_j f(l_1)\rangle = \frac{\partial}{\partial g_{1j}} \frac{\sum_\sigma \sigma_j f(l_1(\sigma)) \exp(\frac{\beta}{\sqrt{N}} \sum_{r<s} g_{rs}\sigma_r\sigma_s)}{\sum_r \exp(\frac{\beta}{\sqrt{N}} \sum_{r<s} g_{rs}\sigma_r\sigma_s)}$$

$$= \frac{\sum_\sigma \left[\sigma_j f'(l_1(\sigma)) \frac{\sigma_j}{\sqrt{N}} \exp(\cdots) + \sigma_j f(l_1(\sigma)) \frac{\beta}{\sqrt{N}} \sigma_1\sigma_j \exp(\cdots)\right]}{\sum_\sigma \exp(\cdots)}$$

$$- \frac{\sum_\sigma f(l_1(\sigma)) \exp(\cdots)}{(\sum \exp(\cdots))^2} \left(\sum_r \frac{\beta}{\sqrt{N}} \sigma_1\sigma_j \exp(\cdots)\right)$$

$$= \frac{\langle f'(l_1)\rangle}{\sqrt{N}} + \frac{\beta}{\sqrt{N}}\langle \sigma_1 f(l_1)\rangle - \frac{\beta}{\sqrt{N}} \langle \sigma_j f(l_1)\rangle \langle \sigma_1\sigma_j\rangle$$

Thus

$$\nu(l_1 f(l_1)) = \frac{N-1}{N}\nu(f'(l_1)) - \frac{\beta(N-1)}{N}\nu(\sigma_1 f(l_1)) - \frac{\beta}{N}\sum_{j=2} N\, \mathbf{E}\langle \sigma_j f(l_1)\rangle\langle\sigma_1\sigma_j\rangle$$

and

$$\langle \sigma_j f(l_1)\rangle\langle\sigma_1\sigma_j\rangle = \langle f(l_1(\sigma^1))\sigma_j^1\sigma_j^2\sigma_1^2\rangle.$$

So

$$\frac{1}{N}\sum_{j=2}^N \langle \sigma_j f(l_1)\rangle\langle\sigma_1\sigma_j\rangle = \langle f(l_1(\sigma^1))\sigma_1^2 R_{1,2}\rangle + O(\frac{1}{N}).$$

Exercise 4: Under $\langle\ \rangle$, what is the conditional expectation of σ_1 given $\sigma_2, \ldots \sigma_N$?

l_1 is a function of $\sigma_2, \ldots \sigma_N$, so $\langle \sigma_1 f(l_1)\rangle = \langle \tanh(\beta l_1) f(l_1)\rangle$. Combining the steps we get,

$$\nu\left(f'(l_1) - (l_1 - \beta\tanh(\beta l_1))\, f(l_1)\right) = O(\frac{1}{\sqrt{N}})$$

For any suitable probability density ρ, if $X \sim \rho$ then for any suitable f we have $\mathbf{E}(f'(X) + \frac{\rho'(X)}{\rho(X)} f(X)) = 0$ (integration by parts), and the converse is also true. Thus, the annealed distribution of l_1 must be close to the distribution with density ρ that satisfies

$$\frac{d}{dx}\log\rho(x) = -(x - \beta\tanh\beta x).$$

This implies

$$\log \rho(x) = \text{ const } - \frac{x^2}{2} + \log \cosh(\beta x)$$

$$\Rightarrow \rho(x) = \text{ Const } \cosh(\beta x) e^{-x^2/2} = \text{ Const } (e^{-(x-\beta)^2/2} + e^{-(x+\beta)^2/2}).$$

Thus, as $N \to \infty$, the annealed distribution of l_1 tends to the symmetric mixture of $N(\beta, 1)$ and $N(-\beta, 1)$.