

Lecture 24

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Continuing with the Sherrington-Kirkpatrick model.

$\forall q \in [0, 1], \forall \beta \geq 0, \forall h \in \mathbb{R}$

$$\mathbf{E} \left(\frac{1}{N} \log Z_N \right) \leq \inf_q \left\{ \log 2 + \mathbf{E} \log \cosh (\beta z \sqrt{q} + h) + \frac{\beta^2 (1 - q)^2}{4} \right\}, \quad (1)$$

where $z \sim N(0, 1)$.

Exercise 1 (From the previous lecture) Show that the right hand side of Equation 1 is minimized when

$$q = \mathbf{E} \tanh^2 (\beta z \sqrt{q} + h).$$

Now,

$$\varphi'(t) = -\frac{\beta^2}{4} \mathbf{E} \left\langle (R_{12} - q)^2 \right\rangle_t + \frac{\beta^2}{4} (1 - q)^2,$$

where $R_{12} = \frac{1}{N} \sum_1^N \sigma_i^1 \sigma_i^2$. This implies that, $\forall 0 \leq q \leq 1$,

$$\frac{1}{N} \mathbf{E} \log Z_N = \varphi(1) \leq \varphi(0) + \frac{\beta^2 (1 - q)^2}{4},$$

and

$$\varphi(0) = \log 2 + \mathbf{E} \log \cosh (\beta z \sqrt{q} + h),$$

where $z \sim N(0, 1)$.

Thus,

$$\frac{1}{N} \mathbf{E} \log Z_N \leq \inf_q \left\{ \log 2 + \mathbf{E} \log \cosh (\beta z \sqrt{q} + h) + \frac{\beta^2 (1 - q)^2}{4} \right\}. \quad (2)$$

From exercise 1, the right hand side of this equation is minimized when $q = \mathbf{E} \tanh^2 (\beta z \sqrt{q} + h)$.

Exercise 2 Show that if $h > 0$, equation (2) has exactly one root.

From the formula for $\varphi'(t)$, we see that approximate equality for equation 2 holds if and only if q is such that $\mathbf{E} \left\langle (R_{12} - q)^2 \right\rangle_t \approx 0$ for $0 \leq t \leq 1$.

Conversely, if $\mathbf{E} \langle (R_{12} - q)^2 \rangle_t \approx 0, \forall 0 \leq t \leq 1$ holds, then we must have approximate equality in Equation 2. Therefore, q must satisfy $q = \mathbf{E} \tanh^2(\beta z \sqrt{q} + h)$.

Let us take $q = q(\beta, h)$ such that $q = \mathbf{E} \tanh^2(\beta z \sqrt{q} + h)$.

First, we observe that at $t = 0$, the coordinates are independent under $\langle \cdot \rangle_0$.

$$\begin{aligned} \langle \sigma_i \rangle_0 &= 1 \cdot \mathbf{P}(\sigma_i = 1) + (-1) \cdot \mathbf{P}(\sigma_i = -1) \\ &= \tanh(\beta z_i \sqrt{2} + h) \\ \langle \sigma_i^1 \sigma_i^2 \rangle_0 &= \langle \sigma_i^1 \rangle_0 \langle \sigma_i^2 \rangle_0 \\ &= \langle \sigma_i \rangle_0^2 \\ &= \tanh^2(\beta z_i \sqrt{q} + h) \end{aligned}$$

Here, σ^1 and σ^2 are i.i.d. from the Gibbs measure, and

$$R_{12} = \frac{1}{N} \sum_{i=1}^N \sigma_i^1 \sigma_i^2 \approx \langle R_{12} \rangle_0 = \frac{1}{N} \sum_{i=1}^N \tanh^2(\beta z_i \sqrt{q} + h).$$

Since $z_1, z_2, \dots, z_N \stackrel{i.i.d}{\sim} N(0, 1)$,

$$\frac{1}{N} \sum_{i=1}^N \tanh^2(\beta z_i \sqrt{q} + h) \approx \mathbf{E} \tanh^2(\beta z_i \sqrt{q} + h) = q.$$

Thus, under $\langle \cdot \rangle_0$, we have $R_{12} \approx q$ with high probability.

In fact, for $\lambda < \frac{1}{2}$,

$$\nu_0 \left(\exp(\lambda N (R_{12} - q)^2) \right) \leq \frac{1}{\sqrt{1 - 2\lambda}}. \quad (3)$$

Exercise 3 Derive equation (3).

Latala's Proof of the concentration of the overlap

Theorem 4 If $\beta < \frac{1}{2}$, for $2s < 1 - 4\beta^2$, $\nu = \nu_1$

$$\nu \exp(sN(R_{12} - q)^2) \leq \frac{1}{\sqrt{1 - 2s - 4\beta^2}}.$$

Thus,

$$\nu (R_{12} - q)^{2k} \leq \frac{(Ck)^k}{N^k}.$$

Proof: Take any function $f = f(\sigma^1, \sigma^2)$ of a pair of spin configurations.

Claim

$$\begin{aligned} \frac{d}{dt} v_t(f) &\equiv v'_t(f) \\ &= \frac{N\beta^2}{2} \left[v_t \left((R_{12} - q)^2 f \right) - 2v_t \left((R_{34} - q)^2 f \right) - 2v_t \left((R_{23} - q)^2 f \right) + 3v_t \left((R_{34} - q)^2 f \right) \right] \end{aligned}$$

If $f \geq 0$ everywhere,

$$\frac{d}{dt} v_t(f) \leq \frac{N\beta^2}{2} \left[v_t \left((R_{12} - q)^2 f \right) + 3v_t \left((R_{34} - q)^2 f \right) \right].$$

Taking $f = (R_{12} - q)^{2k}$,

$$v'_t (R_{12} - q)^{2k} \leq \frac{N\beta^2}{2} \left[v_t (R_{12} - q)^{2k+2} + 3v_t \left((R_{34} - q)^2 (R_{12} - q)^{2k} \right) \right].$$

Now,

$$\begin{aligned} v_t \left((R_{34} - q)^2 (R_{12} - q)^{2k} \right) &\leq \left(v_t (R_{34} - q)^{2k+2} \right)^{\frac{k}{k+1}} \left(v_t (R_{12} - q)^{2k+2} \right)^{\frac{1}{k+1}} \\ &= v_t (R_{12} - q)^{2k+2}. \end{aligned}$$

Combining, we get

$$v'_t (R_{12} - q)^{2k} \leq 2N\beta^2 v_t (R_{12} - q)^{2k+2}.$$

Multiplying both sides by $\frac{\lambda^k N^k}{k!}$ and summing over k , we have

$$v'_t \left(\exp \left(\lambda N (R_{12} - q)^2 \right) \right) \leq 2N\beta^2 v_t \left((R_{12} - q)^2 e^{\lambda N (R_{12} - q)^2} \right).$$

It follows that

$$\frac{d}{dt} v_t \exp \left((\lambda - 2t\beta^2) N (R_{12} - q)^2 \right) \leq 0.$$

In particular,

$$v_1 \left(\exp \left\{ (\lambda - 2\beta^2) N (R_{12} - q)^2 \right\} \right) \leq v_0 \left(e^{\lambda N (R_{12} - q)^2} \right).$$

So, for $\lambda = s + 2\beta^2 < \frac{1}{2}$, we get our results: For $\beta < \frac{1}{2}$, for $2s < 1 - 4\beta^2$,

$$v \exp \left(s N (R_{12} - q)^2 \right) \leq \frac{1}{\sqrt{1 - 2s - 4\beta^2}}.$$

□