STAT C206A / MATH C223A : Stein's method and applications 1

Lecture 22

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Repetition of the setup for the Sherrington Kirkpatrick model:
A so called disorder $(g_{i,j})_{i \le j \le N}$ is drawn from an i.i.d. $N(0,1)$ distribution, conditional on g, $\sigma = (\sigma_1, \dots, \sigma_N)$ has density proportional to

$$\exp\left(\frac{\beta}{\sqrt{N}} \sum g_{ij}\sigma_i\sigma_j + h \sum \sigma_i\right)$$

Definition 1 (Overlap)**.** *Pick σ^1, σ^2 independently from the Gibbs-measure (i.e. the conditional distribution of σ given g). The overlap is defined as the random variable*

$$R_{12} = \frac{1}{N}\sum_i \sigma_i^1 \sigma_i^2$$

Proposition 1. $\exists \beta_0 > 0$*: if $\beta < \beta_0$ then "with high probability" $R_{12} \simeq q$ where $q = q(\beta, h)$ solves*

$$q = E[\tanh^2(\beta z \sqrt{q} + h)]$$

where $z \sim N(0,1)$. More precisely:

$$E\left\langle (R_{12} - q)^{2k} \right\rangle = \frac{C(k)}{N^k} \ \forall\, k, N$$

(Recall: $\langle \cdot \rangle$ denotes conditional expectation given g)

Corollary 1. *Let*

$$RS(\beta, h) := \log 2 + E[\log\cosh(\beta z \sqrt{(}q) + h)] + \frac{\beta^2(1-q)^2}{4}$$

where q is defined as above. Then, for $\beta \le \beta_0$,

$$\frac{\log Z_N}{N} \to RS(\beta, h)$$

in probability as $N \to \infty$, where $Z_N(\beta, h, g)$ denotes the normalizing constant of the Gibbs measure.

The proof of this Corollary proceeds in two steps:

1. Show that
$$\frac{\log Z_N}{N} - E\left[\frac{\log Z_N}{N}\right] \to 0$$
in probability. This is the easier part.

2. Show that
$$E\left[\frac{\log Z_N}{N}\right] \to RS(\beta, h)$$
This latter step is in turn subdivided into two parts: Showing that the lim sup of the left hand side is bounded by the right hand side (which is somewhat easier, and holds for all β, h), and showing inversely that the lim inf is bigger than the right hand side, which only is true for a certain range of β, h.

1 The Gaussian Poincare inequality

1. In 1 dimension: If $Z \sim N(0,1)$ then for all continuous f we have $Varf(Z) \leq E[f'(Z)^2]$

2. If $Z_1, \ldots, Z_n$ i.i.d. $N(0,1)$, then $Varf(Z_1, \ldots, Z_n) \leq \sum E[\left(\frac{\partial f}{\partial z_i}\right)^2]$

Proof:
$1 \to 2$: If X, Y iid then $Varf(x) = \frac{1}{2}E[(f(x) - f(y))^2]$. By Efron-Stein

$$Varf(Z_1, \ldots, Z_n) \leq$$
$$\frac{1}{2}\sum E[(f(Z_1, \ldots, Z_n) - f(Z_1, \ldots, Z_{i-1}, Z_i', Z_{i+1}, \ldots, Z_n))^2] =$$
$$\sum E[Varf(Z_1, \ldots, Z_n|_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n)] \leq \sum E[\left(\frac{\partial f}{\partial z_i}\right)^2]$$

where the last inequality follows from 1.
1, by Stein's method: Given an absolutely continuous f find g such that $g'(x) - xg(x) = f(x) - Ef(Z)$. W.l.o.g. we assume $Ef(Z) = 0$. Then

$$Varf(Z) = Ef(Z)^2 = E((g'(Z) - Zg(Z))f(Z)) = -Eg(Z)f'(Z)$$

(The last equality follows from integration by parts). This implies by Cauchy Schwartz

$$Ef(Z)^2 \leq \sqrt{Ef'(Z)^2 Eg(Z)^2}$$

Now $f'(x) = \frac{\partial}{\partial x}(g' - xg) = g'' - xg' - g$ Hence

$$Ef(Z)^2 = -E(g(Z)(g''(Z) - Zg'(Z) - g(Z))) = E(g'(Z)^2 + g(Z)^2)$$

Thus $Eg(Z)^2 \leq Ef(Z)^2$. Combining we get the result. This generalizes: If $v : R \to R$ is strictly convex and X is a R.V. with density proportional to $\exp(-v(x))$, then

$$Var f(X) \leq E\left(\frac{f'(X)^2}{v''(X)}\right)$$

Proof: find a g such that $g' + \frac{v'}{v} g = f - Ef$.
Exercise: More generally, If $v : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and X is a R.V. with density proportional to $\exp(-v(x))$, then for all absolutely continuous $f : \mathbb{R}^n \to \mathbb{R}$:

$$Var f(X) \leq E\left[(\nabla f(X))^t (Hess\, V(X))^{-1} \nabla f(X)\right]$$

A deep generalization of this is the so called Helffer-Sjöstrand machinery.

Now, $\frac{\log Z_N}{N}$ is a function of g, hence

$$Var\left(\frac{\log Z_N}{N}\right) \leq \sum_{i<j} E\left(\frac{\partial}{\partial g_{ij}} \frac{\log Z_N}{N}\right)^2$$

Now,

$$\frac{\partial}{\partial g_{ij}} \frac{\log Z_N}{N} = \frac{1}{N Z_N} \frac{\partial Z_N}{\partial g_{ij}} =$$

$$\frac{1}{N Z_N} \sum_{\sigma} \frac{\beta}{\sqrt{N}} \sigma_i \sigma_j \exp\left(\frac{\beta}{\sqrt{N}} \sum g_{ij} \sigma_i \sigma_j + h \sum \sigma_i\right) = \frac{\beta}{N^{3/2}} \langle \sigma_i \sigma_j \rangle .$$

Hence

$$\sum_{i<j} E\left(\frac{\partial}{\partial g_{ij}} \frac{\log Z_N}{N}\right)^2 = \frac{1}{N^3} E \sum_{i<j} \langle \sigma_i \sigma_j \rangle^2 \leq \frac{1}{2N}.$$

It follows that

$$Var\left(\frac{\log Z_N}{N}\right) \leq \frac{1}{2N}$$

This also shows

$$\begin{aligned} Var\left(\frac{\log Z_N}{N}\right) &\leq \frac{\beta^2}{2N^3} E \sum_{i<j} \langle \sigma_i \sigma_j \rangle^2) \\ &= \frac{\beta^2}{2N^3} E \left\langle \sum_{i<j} \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 \right\rangle \\ &= \frac{\beta^2}{2N} E \left\langle R_{12}^2 \right\rangle \end{aligned}$$

where σ^1, σ^2 are i.i.d. draws from the Gibbs measure. We will show later that when $\beta < 1$ and $h = 0$, the above bound is of order N^{-2}.