Lecture 21

Lecture date: Oct 15, 2007 *Scribe: Chris Haulk*

Exercise

Let $X = (X_{i,j})_{1 \le i < j \le n}$ denote the Erdös-Renyi random graph $G(n, p)$. Let

$$f(X) = \#\text{triangles}(X) - E(\#\text{triangles}(X)).$$

Obtain concentration inequalities for $f(X)$ by explicitly constructing an antisymmetric F and an exchangeable pair (X, X') such that

$$E[F(X, X')|X] = f(X).$$

and using part (b) the concentration theorem (lecture 12, Sept 24). Hints: independently regenerate a random edge. f is boolean, so it's a polynomial: use polynomials. Thoughts: this will probably give sharp results when p is fixed, $n \uparrow \infty$, but will probably not give sharp results when $p \downarrow 0, n \uparrow \infty$. A serious research problem would be to use part (c) of the theorem and optimize over k to get the correct tail bound in this latter case[1].

1 Spin Glasses: Sherrington-Kirkpatrick Model

This lecture will cover definitions concerning the SK model of spin glasses and state results to be proven in later lectures. Let $g = (g_{ij})_{1 \le i < j \le N}$ be a collection of iid $N(0, 1)$ random variables. Collectively, these random variables are known as the *disorder*. Given the disorder, the spin configuration $\sigma = (\sigma_1, ..., \sigma_N) \in \{0, 1\}^N$ follows a Gibbs distribution with conditional density proportional to

$$\exp\left(\frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N} \sigma_i\right)$$

Let

$$Z_N(\beta, h, g) = \sum_{\sigma \in \{-1,1\}^N} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N} \sigma_i\right)$$

[1]Optimal results are known only up to logarithmic factors, for details check Kim and Vu 2004, *Divide and conquer martingales and the number of triangles in a random graph*, or Janson, Oleszkiewicz, and Rucinski 2004, *Upper tails for subgraph counts in random graphs.*

be the normalizing constant for this Gibbs measure. For any function f of the spins σ and the disorder g, the *quenched law* (or distribution) of $f(g,\sigma)$ is the conditional distribution of f *given* g. The *annealed law* is the unconditional expectation of f. Usually, the quenched expectation of a function $f(g,\sigma)$ is denoted by $\langle f\rangle$:

$$\langle f\rangle = (Z_N(\beta,h,g))^{-1}\sum_{\sigma\in\{-1,1\}^N} f(g,\sigma)\exp\left(\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right).$$

Then the annealed expectation is just $E\langle f\rangle$, which will often be shortened to $\nu(f)$.

Suppose we generate two independent configurations $\sigma^1,\sigma^2\in\{-1,1\}^N$ from the same Gibbs measure. That is, sample g, and then take two (conditionally) independent samples σ^1,σ^2 from the distribution of spins given g. The overlap between σ^1 and σ^2 is defined to be

$$R_{1,2} = \frac{1}{N}\sum_{i=1}^N \sigma_i^1\sigma_i^2.$$

Let $p_N(\beta,h) = N^{-1}E(\log Z_N(\beta,h))$.

Theorem 1 *Replica-symmetric solution of the S-K model.*

There is $\beta_0>0$ *such that* $\forall\beta\in(0,\beta_0)$, $\forall h$

$$\lim_{N\to\infty} p_N(\beta,h) = \log(2) + E\log\cosh(\beta z\sqrt{q}+h)) + \frac{\beta^2(1-q)^2}{4}$$

where $z\sim N(0,1)$ *and* $q=q(\beta,h)$ *is the unique solution of* $q = E[\tanh^2(\beta z\sqrt{q}+h)]$.

$$\frac{\log Z_N}{N} - p_N(\beta,h) \xrightarrow{P} 0$$

(*Note that this is* not *equal to* $\lim_{N\to\infty}\log(\frac{E(Z_N)}{N})$.)

2 History Lesson

Sherrington and Kirkpatrick claimed that the statement above was true for *all* β,h, however it soon became clear that this is false. Talagrand proved the theorem above in 1998 (see also Shcherbina 1997). Parisi conjectured a "broken replica symmetry solution" - this was recently proven to be correct. If $h=0$, the model is more tractable. In that case,

$$\lim_{N\to\infty} p_N(\beta,h) = \frac{\beta^2}{4} + \log(2) \qquad (0\le\beta<1)$$

and there is a phase transition at $\beta = 1$. This was derived by Aizenmann, Lebowitz and Ruelle in 1987. One outstanding and important conjecture is that the replica symmetric solution holds for all (β, h) satisfying

$$\beta^4 E\left[\frac{1}{\cosh^4(\beta z\sqrt{q} + h)}\right] < 1$$

where q and z are as before. The boundary of this region is known as the Almeida-Thouless line (AT line). Talagrand showed that the replica-symmetric solution is invalid beyond the AT line.