Lecture 19

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1 Tsunády's lemma for a given total sum

Let $\epsilon_1, ..., \epsilon_n$ be i.i.d. symmetric random variables taking values in $\{-1, 1\}$ and let $S_n = \sum_{i=1}^n \epsilon_i$. Let π be a uniformly random permutation, and let $S_k = \sum_{i=1}^k \epsilon_{\pi(i)}$ and $W_k = S_k - \frac{kS_n}{n}$.

Lemma 1 (Lemma 3.8 in handout) *There exist universal constants $c > 0$ and $\theta_0 > 0$ such that for any n, any value of S_n, and any fixed k such that $\frac{n}{3} \leq k \leq \frac{2n}{3}$, it is possible to construct W_k and Z_k (where $Z_k \sim N(0, k(n-k)/n)$) on the same probability space such that for any θ with $|\theta| < \theta_0$*

$$\mathbf{E}[\exp(\theta|W_k - Z_k|)] \leq \exp\left(1 + \frac{c\theta^2 S_n^2}{n}\right).$$

Proof: Fix k, and for now denote W_k by W. Define $\widetilde{W} = W + Y$ where Y is a uniform random variable on $[-1, 1]$ and is independent of all of the other random variables we'll use. We would like to show that

$$\mathbf{E}(\widetilde{W}\varphi(\widetilde{W})) = \mathbf{E}(T\varphi'(\widetilde{W})) \tag{1}$$

for any Lipschitz function φ and

$$T = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=k+1}^{n} a_{ij} + (1 - Y^2)/Z \tag{2}$$

where

$$a_{ij} = 1 - \epsilon_{\pi(i)}\epsilon_{\pi(j)} - (\epsilon_{\pi(i)} - \epsilon_{\pi(j)})Y. \tag{3}$$

We can expect that $T \approx k(n-k)/n$. In fact, it can be shown that if $\sigma^2 = k(n-k)/n$ then

$$\frac{(T - \sigma^2)^2}{\sigma^2} \leq C\left(\frac{S_k^2}{k} + \frac{S_n^2}{n} + 1\right) \tag{4}$$

where C is a universal constant.

By Lemma 3.6, we know that

$$\mathbf{E}[\exp(\theta S_k^2/k)] \leq A\exp(c\theta S_n^2/n) \tag{5}$$

where A and c are constants.

The proof can now be completed using this and Lemma 3.4. Now let's show (1).

Let us write

$$W = S_k - \frac{kS_n}{n} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=k=1}^{n}(\epsilon_{\pi(i)} - \epsilon_{\pi(j)}) \tag{6}$$

and notice that if we fix i and j and condition on $(\pi(l))_{l\notin\{i,j\}}$, then the conditional expectation of $(\epsilon_{\pi(i)} - \epsilon_{\pi(j)})$ must be zero.

So, we fix i and j such that $i \le k \le j$ and condition on $(\pi(l))_{l\notin\{i,j\}}$. Denote S_n by S and let $S^- = \sum_{l\notin\{i,j\}} \epsilon_{\pi(l)}$. Let $\mathbf{E}^-$ denote conditional expectation, and consider

$$\mathbf{E}^-[(\epsilon_{\pi(i)} - \epsilon_{\pi(j)})\varphi(\widetilde{W})]. \tag{7}$$

Case 1: if $S \neq S^-$, that is, $\epsilon_{\pi(i)} = \epsilon_{\pi(j)}$, then (7) $= 0$.

Case 2: if $S = S^-$, that is, $\epsilon_{\pi(i)} = -\epsilon_{\pi(j)}$, then let $X = \frac{1}{2}(\epsilon_{\pi(i)} - \epsilon_{\pi(j)}) = \epsilon_{\pi(i)}$. So, $W = W^- + X$ where

$$W^- = \sum_{l=1}^{k}\epsilon_{\pi(l)} - \epsilon_{\pi(i)} - \frac{kS}{n}.$$

Now, using Lemma 3.7 we obtain

$$\begin{aligned}
\mathbf{E}^-[(\epsilon_{\pi(i)} - \epsilon_{\pi(j)})\varphi(\widetilde{W})] &= \mathbf{E}^-[2X\varphi(W^- + X + Y)] \\
&= 2\mathbf{E}^-[(1 - XY)\varphi'(\widetilde{W})] \\
&= \mathbf{E}^-[(2 - (\epsilon_{\pi(i)} - \epsilon_{\pi(j)})Y)\varphi'(\widetilde{W})].
\end{aligned}$$

If we define a_{ij} as in (3) above, then

$$a_{ij} = \begin{cases} 2 - (\epsilon_{\pi(i)} - \epsilon_{\pi(j)})Y & \text{if } S = S^-; \\ 0 & \text{if } S \neq S^-. \end{cases}$$

Finally we obtain

$$\mathbf{E}^-[(\epsilon_{\pi(i)} - \epsilon_{\pi(j)})\varphi(\widetilde{W})] = \mathbf{E}^-(a_{ij}\varphi'(\widetilde{W})] \tag{8}$$

and replace $\mathbf{E}^-$ by $\mathbf{E}$ by taking expectations on both sides. If we again apply Lemma 3.7 we obtain (1). □

Exercise 2 *Prove Lemma 3.5.*

Lemma 3 (Lemma 3.9 in handout) *For any n and any possible value of S_n we can construct $W_0, W_1, ..., W_n$ and $Z_0, Z_1, ..., Z_n$ where the Z_i's are jointly Gaussian with mean zero and*

$$Cov(Z_i, Z_j) = \frac{(i \vee j)(n - (i \vee j))}{n}$$

such that for all $\lambda \leq \lambda_0$,

$$\mathbf{E}[\exp(\lambda \max_{i \leq n} |W_i - Z_i|)] \leq \exp\left(C \log n + \frac{K\lambda^2 S_n^2}{n}\right) \tag{9}$$

where C, K, and λ_0 are universal constants.

The proof is by induction: use $\exp(x \vee y) \leq e^x + e^y$. *To be continued.*