

Lecture 11

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Scribe: Maximilian Kasy

For E the set of edges in the lattice \mathbb{Z}^2 , let $(\omega_e)_{e \in E}$ be i.i.d. nonnegative edge weights. Define

$$t_n := \inf \left\{ \sum_{e \in P} \omega_e : P \text{ is a path from } (0,0) \text{ to } (n,0) \right\}$$

Then $E(t_{n+m}) \leq E(t_n) + E(t_m) \forall n, m$, as shown last time.

Lemma 1 (Subadditive Lemma). *If $\{a_n\}$ is a sequence of real numbers, such that $a_{n+m} \leq a_n + a_m \forall n, m$, then*

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}$$

In particular the limit exists.

Proof: Fix $k \geq 1$. Take any n and choose m such that $n = mk + r$ with $0 \leq r < k$. Then $a_n \leq ma_k + a_r$. It follows that $\limsup \frac{a_n}{n} \leq \frac{a_k}{k}$, hence

$$\limsup \frac{a_n}{n} \leq \inf \frac{a_k}{k} \leq \liminf \frac{a_n}{n}$$

which concludes the proof.

Now suppose for the rest of this lecture that $E(\omega_e^2) < \infty$. It follows from the lemma that $\exists 0 \leq \mu < \infty$ such that $E(t_n)/n \rightarrow \mu$ as $n \rightarrow \infty$.

Kersten showed: If $E(\omega_e^2) < \infty$ and $P(\omega_e = 0) < p_c(d)$, where $p_c(d)$ is the critical probability for bond percolation in \mathbb{Z}^d , then $\mu > 0$.

We want to show: $\frac{t_n}{n} \rightarrow \mu$ in probability. Assume now

$$\exists a > 0 \text{ s.t. } P(\omega_e > a) = 1 \tag{1}$$

We will show an inequality of the form $Var(t_n) \leq Cn$. Let l_n be the number of edges in a shortest minimal-weight path. Under assumption 1, $l_n \leq \frac{t_n}{a}$. By the argument of last lecture,

$$Var(t_n) \leq \frac{1}{2} \sum_{e \in E} E(t_n(\omega) - t_n(\omega^{(e)}))^2$$

where $\omega_u^{(e)} = \omega_u$ if $u \neq e$ and $\omega_e^{(e)} = \omega'_e$ where the latter is an independent copy of ω_e . By symmetry, then,

$$Var(t_n) \leq \sum_{e \in E} E \left[(t_n(\omega) - t_n(\omega^{(e)}))^2 \mathbf{1}(\omega_e \leq \omega'_e) \right] =: \otimes \text{ (say)}.$$

Now, if $\omega_e \leq \omega'_e$ and $t_n(\omega) \neq t_n(\omega^{(e)})$, then it is easy to argue that e must be in every minimal path from $(0, 0)$ to $(n, 0)$ under the configuration ω , and $(t_n(\omega) - t_n(\omega^{(e)}))^2 \leq (\omega'_e - \omega_e)^2 \leq (\omega'_e)^2$. Indeed, if the length of all minimal paths increase after increasing ω_e to ω'_e , then e must belong to all minimal paths in ω , and we necessarily have $t_n(\omega^{(e)}) \leq t_n(\omega) + \omega'_e - \omega_e$. Hence

$$\begin{aligned} \otimes &\leq \sum_e E((\omega'_e)^2 \mathbf{1}(e \in \text{every minimal path in } \omega)) \\ &= \sum_e E((\omega'_e)^2) E(\mathbf{1}(e \in \text{every minimal path in } \omega)) \\ &= E(\omega_e^2) E\left(\sum_e \mathbf{1}(e \in \text{every minimal path in } \omega)\right) \\ &\leq E((\omega_e)^2) E(l_n) \leq Cn \end{aligned}$$

by assumption 1 and the fact that $E(t_n)/n \rightarrow \mu$. This concludes the argument.

It has been conjectured that actually $C_1 n^{2/3} \leq \text{Var}(t_n) \leq C_2 n^{2/3}$ in \mathbb{Z}^2 , where C_1 and C_2 are positive constants depending on the distribution of ω_e . The best known lower bound is $C \log n$.

Let us now state two standard tools for proving concentration inequalities.

Theorem 1 (Azuma-Hoeffding inequality). *Suppose X_1, \dots, X_n are martingale differences with respect to a filtration $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ and there exist constants a_i, b_i such that $a_i \leq X_i \leq b_i$ almost surely. Then*

$$P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i \geq t\right) \leq \exp\left(-\frac{2t^2}{\sum (b_i - a_i)^2}\right)$$

Theorem 2 (Bounded differences inequality). *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is such that there are constants c_1, \dots, c_n such that $|f(x) - f(y)| \leq c_i$ whenever x and y differ only in the i th coordinate. Suppose X_1, \dots, X_n are independent random variables. Then*

$$P(f(X_1, \dots, X_n) - E(f(X_1, \dots, X_n)) \geq t) \leq \exp\left(-\frac{2t^2}{\sum c_i^2}\right)$$

Sketch of a proof: Put $Y_i = E(f(X)|X_1, \dots, X_i) - E(f(X)|X_1, \dots, X_{i-1})$ and apply the previous theorem.