Statistics 150 (Stochastic Processes): Midterm Exam, Spring 2010. J. Pitman, U.C. Berkeley.

Convention throughout: $\min \emptyset = \infty$.

1. Let $X_0, Y_1, Y_2, \ldots$ be independent random variables, X_0 with values in $\{0, 1, 2, \ldots\}$ and each Y_i an indicator variable with values in $\{0, 1\}$ and $\mathbb{P}(Y_i = 1) = 1/i$ for each $i = 1, 2, \ldots$. For $n = 1, 2, \ldots$ let

$$X_{n+1} := \max\{k : 1 \le k < X_n \text{ and } Y_k = 1\} \text{ if } X_n > 1$$

and $X_{n+1} := 0$ if $X_n \le 1$. Explain why (X_n) is a Markov chain, and describe its state space and transition probabilities.

2. For $Y_1, Y_2, \ldots$ as in the previous question, let $T_0 := 0$ and for $n = 1, 2, \ldots$ let

$$T_n := \min\{k : k > T_{n-1} \text{ and } Y_k = 1\}$$

Explain why (T_n) is a Markov chain, and describe its state space and its transition probabilities.

3. Let X, Y, Z be random variables defined on a common probability space, each with a discrete distribution. Explain why the function $\psi(x) := \mathbb{E}(Y \mid X = x)$ is characterized by the property

$$\mathbb{E}(Y g(X)) = \mathbb{E}[\psi(X) g(X)]$$

for every bounded function g whose domain is the range of X. Use this characterization of $E(Y|X)$ to verify the formula

$$\mathbb{E}(E(Y|X) \mid f(X)) = \mathbb{E}[Y \mid f(X)]$$

for every function f whose domain is the range of X, and the formula

$$\mathbb{E}(E(Y|X, Z) \mid X) = \mathbb{E}[Y|X].$$

4. Suppose that a sequence of random variables $X_0, X_1, \ldots$ and a function f are such that

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n) = f(X_n)$$

for every $n = 0, 1, 2, \ldots$. Explain why this implies

$$\mathbb{E}(f(X_{n+1}) \mid f(X_0), \ldots, f(X_n)) = f(X_n)$$

Give an example of such an f which is not constant for (X_n) a $p \uparrow, 1-p \downarrow$ random walk on the integers.

5. Let $S := X_1 + \cdots + X_N$ be the number of successes and $F := N - S$ the number of failures in a Poisson(μ) distributed random number N of Bernoulli trials, where given $N = n$ the $X_1, \ldots, X_n$ are independent with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$ for some $0 \le p \le 1$. Derive the joint distribution of S and F. How can the conclusion be generalized to multinomial trials?

6. Let $\mathbb{P}_i$ govern a $p \uparrow, 1 - p \downarrow$ walk (S_n) on the integers started at $S_0 = i$, with $p > q$. Let

$$f_{ij} := \mathbb{P}_i(S_n = j \text{ for some } n \ge 1).$$

Use results derived in lectures and/or the text to present a formula for f_{ij} in each of the two cases $i > j$ and $i < j$. Deduce a formula for f_{ij} for $i = j$.

7. Let $\mathbb{P}_i$ govern (X_n) as a Markov chain starting from $X_0 = i$, with finite state space S, and transition matrix P which has a set of absorbing states B. Let $T := \min\{n \ge 1 : X_n \in B\}$ and assume that $\mathbb{P}_i(T < \infty) = 1$ for all i. Derive a formula for

$$\mathbb{P}_i(X_{T-1} = j, X_T = k) \text{ for } i, j \in B^c \text{ and } k \in B$$

in terms of the matrices $W := (I - Q)^{-1}$ and R, where Q is the restriction of P to $B^c \times B^c$ and R is the restriction of P to $B^c \times B$.

8. In the same setting, let $f_{ij} := \mathbb{P}_i(X_n = j \text{ for some } n \geq 1)$. For $i, j \in B^c$, find and explain a formula for f_{ij} in terms of W_{ij} and W_{jj}.

9. In the same setting, let $\phi_i(s)$ denote the probability generating function of T for the Markov chain started in state i. Derive a system of equations which could be used to determine $\phi_i(s)$ for all $i \in S$.

10. Let X be a non-negative integer valued random variable with probability generating function $\phi(s)$ for $0 \leq s \leq 1$. Let N be independent of X with the geometric(p) distribution $\mathbb{P}(N = n) = (1-p)^n p$ for $n = 0, 1, 2, \ldots$, where $0 < p < 1$. Find a formula in terms of ϕ and p for $\mathbb{P}(N < X)$.

11. Let X be a non-negative random variable with usual probability generating function $\phi(s)$ for $0 \leq s \leq 1$. Define the *tail probability generating function* $\tau(s)$ by

$$\tau(s) := \sum_{n=1}^{\infty} \mathbb{P}(X \geq n) s^n$$

Use the identity

$$\mathbb{P}(X = n) = \mathbb{P}(X \geq n) - \mathbb{P}(X \geq n+1)$$

to help derive a formula for $\tau(s)$ in terms of s and $\phi(s)$ for $0 \leq s < 1$. Discuss what happens for $s = 1$.

12. Consider a random walk on the 3 vertices of a triangle labeled clockwise by $0, 1, 2$. At each step, the walk moves clockwise with probability p and counter-clockwise with probability q, where $p + q = 1$. Let P denote the transition matrix. Observe that

$$P^2(0,0) = 2pq; \quad P^3(0,0) = p^3 + q^3; \quad P^4(0,0) = 6p^2q^2.$$

Derive a similar formula for $P^5(0,0)$.

13. A branching process with Poisson(λ) offspring distribution started with one individual has extinction probability p with $0 < p < 1$. Find a formula for λ in terms of p.

14. Suppose (X_n) is a Markov chain with state space $\{0, 1, \ldots, b\}$ for some positive integer b, with states 0 and b absorbing and no other absorbing states. Suppose also that (X_n) is a martingale. Evaluate

$$\lim_{n \to \infty} \mathbb{P}_a(X_n = b)$$

and explain your answer carefully.