Stat 150 Midterm, J.P., Spring 2006. Student name in CAPITALS: ______________________

A. Warmup. Little explanation required. Just apply results from class notes or text.

1. A Poisson(λ) number of dice are rolled. Let N_i be the number of times face i appears among these dice. Describe the joint distribution of N_1 and N_2.
2. A random walk starting at 3 moves up 1 at each step with probability 2/3 and down 1 with probability 1/3. What is the probability that the walk reaches 10 before 0?
3. For the same random walk, what is the expected number of steps until reaching either 0 or 10?
4. Let T_r be the number of tosses until the rth H in independent coin tosses with p the probability of H on each toss. Write down the probability generating function of T_r.
5. Let U_r be the number of tosses until the pattern of "HTHT....HT" of length $2r$ appears (meaning the "HT" is repeated r times in a row). Give a formula for $E(U_r)$.

B. Consider a Markov chain (X_n) with transition matrix P. Let f be a function with numerical values defined on the state space of the Markov chain. Using suitable matrix notation and/or summation notation,

1. For positive integers m and n give a formula for $E(f(X_{n+m}) \mid X_n = k)$
2. Suppose that $f(i) = \sum_j P(i,j)f(j)$ for all states i. What can you then say about the process $f(X_n)$? Explain carefully.
3. Suppose that $f(i) = \sum_j P(i,j)f(j)$ for all states i, that $|f(i)| \leq 5$ for all states i, that there are precisely two absorbing states 0 and b, one or other of which is reached in finite time with probability 1, no matter what the initial state i, and that $f(b) \neq f(0)$. Derive a formula for the probability, starting in state i, that the chain ends up being absorbed in state b.

C. Let Z_n be the number of individuals in the nth generation of a branching process starting with $Z_0 = 1$, with offspring probability generating function $\phi(s) = \sum_{n=0}^{\infty} p_n s^n$. Let N be the least n such that $Z_n = 0$, with $n = \infty$ if no such n. Let M be the least n such that there is some individual in generation n who has no children.

1. For $n = 1, 2, 3, 4$, find expressions for $P(N \geq n)$ in terms of the function ϕ.
2. Let $q_n = P(M \geq n)$. Derive a formula for q_{n+1} involving the function ϕ, q_n, and p_0.
3. For $n = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$ find a formula for

$$P(M \geq n+1 \mid M \geq n, Z_{n+1} = k)$$

   which shows this conditional probability is at most $1 - p_0$.
4. Deduce that $E(M) \leq 1/p_0$.