Statistics 150 (Stochastic Processes): Final Exam, Spring 2009. J. Pitman, U.C. Berkeley.

1. Suppose that a Markov matrix P indexed by a finite set has the property that for each state i:

$$\sum_{j \neq i} P(i,j) = \sum_{j \neq i} P(j,i).$$

a) What does this property imply about the stationary distribution of the Markov chain?
b) Does this property imply that the stationary distribution is unique? If so, sketch a proof, and if not provide a counter-example.

2. Consider three independent Poisson arrival processes $N_i(t), t \geq 0$ with rates λ_i for $i = 1, 2, 3$, all starting at $N_i(0) = 0$. Let T_{31} be the least $t \geq 0$ such that $N_3(t) = 1$, and let $X_i = N_i(T_{31})$ for $i = 1, 2$.
a) Describe the distribution of X_i for each $i = 1, 2$.
b) Describe the joint distribution of X_1 and X_2.
c) Find $E(X_2|X_1)$.

3. Let (X_n) be a Markov chain with state space $\{0, 1, \dots, 2N\}$ for some positive integer N with transition matrix P such that

$$P(i, i-1) = P(i, i+1) = 1/2 \text{ for } 1 \leq i \leq 2N-1,$$
$$P(0, N) = P(2N, N) = 1.$$

a) Write down the equations satisfied by the stationary distribution π for this Markov chain, with special attention to the equations associated with states $0, N$ and $2N$.
b) Show that

$$\pi_0 = \frac{1}{2(N^2+1)}$$

and give explicit expressions for all other π_i.
c) Let T_0 denote the first return time to state 0 given that $X_0 = 0$. Explain why

$$T_0 = Y_1 + \cdots + Y_M$$

for a sequence of independent and identically distributed random variables Y_i with $\mathbb{E}_0(Y_i) = N^2 + 1$ and a random index M which is independent of $Y_1, Y_2, \ldots$. What is the distribution of M?

4. Suppose that the unit interval $[0, 1)$ is broken into $n + 1$ subintervals

$$[0, U_{n,1}), [U_{n,1}, U_{n,2}), \dots, [U_{n,n}, 1)$$

by cutting at each of n points picked independently and uniformly at random from $[0, 1]$, with $U_{n,i}$ the ith of these points arranged in increasing order. For $0 \leq t \leq 1$ let $[t - \delta_t, t + \gamma_t)$ denote the subinterval that contains t.
a) Find a formula for

$$P(\delta_t > x) \text{ for } 0 < x < t$$

b) Find a formula for

$$P(\gamma_t > y) \text{ for } 0 < y < 1-t$$

c) Deduce that the expected length of the interval containing t is

$$\frac{1}{n+1}(2 - (1-t)^{n+1} - t^{n+1})$$

d) Show that if U is a further random point picked uniformly from $[0, 1]$, independently of the n points used to make the cuts, then the expected length of the interval containing U is $\frac{2}{n+2}$.

e) It is known (and you can assume) that the lengths of the $n + 1$ intervals are identically distributed. Use this fact to find the distribution of the length of the interval containing U.

5. Rainfall times at a certain location occur according to a Poisson process with rate λ. Each time it rains, the amount of rain that falls has a uniform distribution on $[0, b]$ independently of times and amounts of all other rainfall. Let R_t be the total amount of rainfall up to time t, and for $0 < a < b$ let $N(a,t)$ be the number of times it rains more than amount a before time t.
   a) What is the distribution of $N(a,t)$?
   b) Calculate $E[R_t \,|\, N(a,t) = n]$.

6. Taxis looking for customers arrive at a taxi station as a Poisson process (rate 1 per minute), while customers looking for taxis arrive as a Poisson process (rate 1.25 per minute). Suppose taxis will wait, no matter how many taxis are in line before them. But customers who arrive to find 2 other customers in line go away immediately. Over the long run, what is average number of customers waiting at the station? [Hint: set up a chain with $\{-2, -1, 0, 1, 2, 3, \ldots\}$ as states.]

7. Suppose a continuous time Markov chain has a state 0 with

$$P_t(0,0) = \sum_{j=1}^{k} a_j e^{-b_j t}$$

   for some finite k. Let H_0 denote the holding time of the chain in state 0 before it jumps to some other state. Find the expected value of H_0.

8. Suppose that battery lifetimes are independent with the gamma(r, λ) distribution, whose density at $t > 0$ is $\Gamma(r)^{-1}\lambda^r t^{r-1} e^{-\lambda t}$, for some $r > 0$ and $\lambda > 0$. In a system requiring one battery, the battery is replaced by a new one as soon as it dies. Let L_t denote the total lifetime (that is current age plus remaining lifetime) of the battery in use at time t. Describe the limit distribution of L_t as $t \to \infty$, and calculate the mean of this limit distribution.

9. Let $(B_t, t \geq 0)$ be a standard Brownian motion. Find the distribution of $B_s + B_t$ for $0 < s < t$.

10. Let S_n be a simple, symmetric random walk with increments of ± 1 started at $S_0 = 0$. Let

$$M_n := \max_{1 \leq k \leq n} S_k.$$

   a) For which particular power p does the limit

$$\lim_{n \to \infty} \mathbb{P}(M_n / n^p \leq 1)$$

   have a value which is neither 0 nor 1?
   b) For this particular p, express the value of the limit as an integral.