ARCSINE LAWS AND INTERVAL PARTITIONS DERIVED FROM A STABLE SUBORDINATOR

JIM PITMAN *and* MARC YOR

[Received 4 January 1991—Revised 3 September 1991]

ABSTRACT

Lévy discovered that the fraction of time a standard one-dimensional Brownian motion B spends positive before time t has arcsine distribution, both for t a fixed time when $B_t \neq 0$ almost surely, and for t an inverse local time, when $B_t = 0$ almost surely. This identity in distribution is extended from the fraction of time spent positive to a large collection of functionals derived from the lengths and signs of excursions of B away from 0. Similar identities in distribution are associated with any process whose zero set is the range of a stable subordinator, for instance a Bessel process of dimension d for $0 < d < 2$.

1. *Introduction*

Let $(B_t, t \geq 0)$ be a Brownian motion on the line, starting at 0. Let $\Gamma_+(t)$ be the time B spends above 0 up to time t:

$$\Gamma_+(t) = \int_0^t 1(B_s > 0)\, ds.$$

Lévy [**29**] showed that for each $t > 0$ the variable $\Gamma_+(t)/t$ has the arcsine law:

$$P(\Gamma_+(t)/t \in du) = \frac{du}{\pi\sqrt{(u(1-u))}}, \quad \text{for } 0 < u < 1. \tag{1.a}$$

On the way to this result, Lévy showed that the same arcsine distribution is obtained if the fraction of time spent positive, $\Gamma_+(t)/t$, is considered at the random time $t = T_s$, where $(T_s, s \geq 0)$ is the inverse of the continuous local time process $(S_t, t \geq 0)$ which Lévy associated with the random set of zeros of B. We denote this identity in distribution by

$$\frac{\Gamma_+(t)}{t} \overset{d}{=} \frac{\Gamma_+(T_s)}{T_s}. \tag{1.b}$$

As Lévy noted in the third paragraph of p. 326 of his 1939 paper [**29**], it is remarkable that the same law for the fraction of time spent positive should appear both at a fixed time t when $B_t \neq 0$ almost surely, and for the random time $t = T_s$ when $B_t = 0$ almost surely. Our aim is to expose as best we can what lies behind the identity of laws (1.b). This enables us to extend the identity to a large collection of functionals derived from the lengths and signs of excursions of B away from 0. We find similar identities for a process whose zero set is the range of a stable subordinator. These results are closely related to the multidimensional arcsine laws of Barlow, Pitman and Yor [**2**].

Research supported in part by NSF grants DMS88-01808 and DMS91-07531.

1991 *Mathematics Subject Classification*: primary 60E07; secondary 60G55, 60J30, 60J55, 60J65.

We start by recalling how Lévy derived the arcsine distribution of $\Gamma_+(T_s)/T_s$. He first showed that $(T_s, s \geq 0)$ is an increasing process with independent increments, or *subordinator,* which is stable with index $\alpha = \frac{1}{2}$. There is a one-to-one correspondence between jumps of this subordinator and maximal open intervals during which the Brownian motion is away from zero. From the Poisson character of the jumps of the subordinator, and the fact that each of these jumps contributes to time positive with probability $\frac{1}{2}$, independently of all others, Lévy argued that

$$(\Gamma_+(T_s), T_s) \stackrel{d}{=} (T_{s/2}, T_s), \tag{1.c}$$

so that

$$\frac{\Gamma_+(T_s)}{T_s} \stackrel{d}{=} \frac{T_{s/2}}{T_{s/2} + T^*_{s/2}}, \tag{1.d}$$

where $T^*_{s/2} = T_s - T_{s/2}$ is independent of $T_{s/2}$ with the same stable distribution with index $\frac{1}{2}$. Lévy showed that the distribution of the right-hand variable in (1.d) is arcsine by a two-dimensional integration. This can also be seen more simply as in paragraph (1.1) of [**2**].

Using (1.c) and the formula

$$P(T_s \in dt)/dt = \frac{s}{\sqrt{(2\pi)}} t^{-\frac{3}{2}} e^{-s^2/2t}, \quad \text{where } s > 0,\ t > 0, \tag{1.e}$$

Lévy gave an explicit formula for the density of the conditional distribution

$$P(\Gamma_+(T_s) \in \cdot \mid T_s = t) = P(\Gamma_+(t) \in \cdot \mid B_t = 0,\ S_t = s). \tag{1.f}$$

By integration with respect to the conditional distribution of S_t given $B_t = 0$, he then obtained his famous result for the time spent positive by a Brownian bridge:

(1.g) *the conditional distribution of* $\Gamma_+(t)/t$ *given* $B_t = 0$ *is uniform on* $[0, 1]$.

Finally, Lévy pointed out that the unconditional distribution of $\Gamma_+(t)/t$ could be found by conditioning on the time G_t of the last zero before time t. Lévy showed that G_t/t too has arcsine distribution and that

(1.h) *given* $G_t = g$, *the pre-*g *process* $(B_u, 0 \leq u \leq g)$ *is a Brownian bridge of length* g.

He then derived the arcsine law (1.a) for $\Gamma_+(t)/t$ by applying (1.g) to the bridge in (1.h), and using the fact that independently of what happens before g, on the final zero-free interval of length $1 - g$, the Brownian path is equally likely to go positive or negative.

The most important ingredient in these results of Lévy is the fact that the random zero set Z of the Brownian motion B is the closure of the range of a stable subordinator of index $\alpha = \frac{1}{2}$. Somewhat more generally, suppose B is replaced by a diffusion on the line whose zero set is the closed range of a subordinator $(T_s, s \geq 0)$, which is *stable with index* α, for some $0 < \alpha < 1$, meaning that for every $a > 0$,

$$(T(s), s \geq 0) \stackrel{d}{=} (aT(s/a^\alpha), s \geq 0). \tag{1.i}$$

(We often use notation like $T(s)$ instead of T_s for typographical convenience.) A diffusion B with such an inverse local time process at zero can be constructed as

in [1] using a Bessel process of dimension $d = 2(1 - \alpha)$ for the modulus, and coin tossing for the signs. Lévy's formulae (1.c) and (1.d) adapt at once to this setting to describe the law of $\Gamma_+(T_s)/T_s$. And as shown by a Laplace transform calculation in [2], the identity in law (1.b) between $\Gamma_+(t)/t$ and $\Gamma_+(T_s)/T_s$ continues to hold, even though Lévy's approach sketched above is blocked in this setting by the lack of any analogue of the explicit formula (1.e) for $\alpha \neq \frac{1}{2}$.

Puzzled by why (1.b) should hold so generally, we were led to analyse the *interval partition* generated by the zeros up to time t. By this we mean the collection of lengths of maximal intervals comprising $Z^c \cap (0, t)$, without regard to the order in which these lengths appear. We describe the interval partition by the sequence of ranked lengths of intervals, as in the following theorem:

THEOREM 1.1. *Fix α with $0 < \alpha < 1$. Suppose Z is the closure of the range of a stable subordinator (T_s) of index α. Let $\mathbf{V}(t)$ be the infinite sequence of lengths of the maximal open subintervals of $Z^c \cap (0, t)$, arranged in descending order:*

$$\mathbf{V}(t) = (V_1(t), V_2(t), V_3(t), \ldots) \tag{1.j}$$

where $V_1(t) \geq V_2(t) \geq V_3(t) \geq \ldots$. Then for $t > 0$ and $s > 0$,

$$\frac{\mathbf{V}(t)}{t} \stackrel{d}{=} \frac{\mathbf{V}(T_s)}{T_s}. \tag{1.k}$$

To illustrate this result in the Brownian case, $\alpha = \frac{1}{2}$, suppose that Z is the zero set of the Brownian motion, and each interval of Z^c is given a sign by a fair coin tossing process independent of interval length. So the ith longest interval of $Z^c \cap (0, t)$, that is $V_i(t)$, contributes with probability $\frac{1}{2}$ to the sum of positive interval lengths $\Gamma_+(t)$, independently as i varies and independently of $\mathbf{V}(t)$. The same can be said at the random time T_s instead of the fixed time t. So Lévy's identity (1.b) follows at once from (1.k).

Notice that for a fixed time t, one of the lengths appearing in the sequence $\mathbf{V}(t)$ represents the age A_t of the excursion in progress at time t:

$$A_t = t - G_t, \quad \text{where } G_t = \sup\{Z \cap (0, t)\} \tag{1.m}$$

is the last time in Z before t. More precisely, $A_t = V_{\#(t)}(t)$, where $\#(t) - 1$ is the number of excursions completed before time t with lifetime longer than A_t. As a complement to Theorem 1.1 we show:

THEOREM 1.2. *Conditionally given $\mathbf{V}(t)$, the length of the final interval is picked by length biased sampling. In symbols:*

$$P(\#(t) = j \mid \mathbf{V}(t)) = V_j(t)/t, \quad \text{for } j = 1, 2, \ldots.$$

In contrast to a fixed time t, the random time T_s falls in Z almost surely, so $G_{T(s)} = T_s$ and $A_{T(s)} = 0$ almost surely. This means that every length appearing in $\mathbf{V}(T_s)$ represents the lifetime of a complete excursion. And $\#(T_s)$ is undefined. So despite the equality in distribution of $\mathbf{V}(t)/t$ and $\mathbf{V}(T_s)/T_s$, it is nonsense to substitute the random time T_s instead of the fixed time t in Theorem 1.2. But suppose V_s^* is the length of the complete excursion containing a random time point T_s^*, which given the subordinator is distributed uniformly on the interval

$(0, T_s)$. Then, according to Theorems 1.1 and 1.2, the joint distribution of $\mathbf{V}(t)/t$ and A_t/t is the same as that of $\mathbf{V}(T_s)/T_s$ and V_s^*/T_s.

Kingman [**25**, § 7] considered the distribution of $\mathbf{V}(T_s)/T_s$ for a more general subordinator. In particular, in the case where (T_s) is a gamma process, the law of $\mathbf{V}(T_s)/T_s$ is the well-known Poisson–Dirichlet distribution on the infinite simplex with parameter s. This distribution turns up in a number of different contexts, for example in the asymptotics for the ranked cycle lengths of a random permutation (Shepp and Lloyd [**42**], Vershik and Schmidt [**43**, **44**]), as the distribution of the ranked sizes of atoms in the Dirichlet process prior in Bayesian statistics (Blackwell and McQueen [**5**], Ferguson [**11**]), and in limiting models for the abundances of genes in population genetics and species in mathematical ecology (see, for example, Patil and Taillie [**32**], Hoppe [**17**]). Explicit but rather intractable formulae for features of the Poisson–Dirichlet distribution such as the joint density of the first n components can be found in papers by Watterson [**46**], Ewens [**10**], and some of the other references above. Perman [**34**] gives extensions of these formulae for a class of subordinators including both the stable and gamma cases in terms of the solution of an integral equation. In the stable case the one-dimensional density of the largest component $V_1(T_s)/T_s$ is found quite simply on the interval $(\frac{1}{2}, 1]$, then determined recursively by the integral equation on the intervals $(\frac{1}{3}, \frac{1}{2}]$, $(\frac{1}{4}, \frac{1}{3}]$, and so on. But explicit integration seems possible only in the case where $\alpha = \frac{1}{2}$ and even then only as far as the interval $(\frac{1}{3}, \frac{1}{2}]$. See also Getoor [**14**] and Knight [**26**] for results on the distribution of $V_1(G_t)$ at fixed times t for a general subordinator.

While explicit description of the common finite-dimensional distributions of $\mathbf{V}(T_s)/T_s$ and $\mathbf{V}(t)/t$ in the stable case turns out to be unmanageable, Theorem 1.2 hints that there may be a simple description for the size-based permutation of the values of $\mathbf{V}(T_s)/T_s$, analogous to the following result of McCloskey [**30**] in the Poisson–Dirichlet case:

If T_s has gamma(s) *distribution and $Y_1, Y_2, \ldots$ represents a size-biased permutation of $\mathbf{V}(T_s)/T_s$, then*

$$Y_i = \xi_i \prod_{j=1}^{i-1} (1 - \xi_j), \quad \textit{for } i \geqslant 1, \tag{1.n}$$

where the ξ_i are independent with identical beta($1, s$) *distribution.*

Indeed, Perman [**34**, Corollary 3.19] has applied Theorems 1.1 and 1.2, to derive the following remarkable analogue in the stable case:

If T_s has stable(α) *distribution, then the size-biased permutation of $\mathbf{V}(T_s)/T_s$ is described by* (1.n) *for independent and non-identically distributed ξ_i, where ξ_i has* beta($1 - \alpha, i\alpha$) *distribution.*

Due to Theorem 1.1, (Y_i) constructed as in (1.n) from such beta variables also describes the length-biased permutation of the unit interval partition $\mathbf{V}(t)/t$ associated with the stable(α) subordinator, for any $t > 0$. These and other results concerning size-biased sampling of the jumps of a subordinator are studied by Perman, Pitman and Yor in [**35**].

If we now think of Z as the zero set of an underlying Brownian motion or Bessel process, Theorems 1.1 and 1.2 invite interpretation at the level of some kind of path transformation. In the case of classical identities in distribution related to arcsine laws for random walks, this is achieved by the Sparre–Andersen transformations described by Feller [**12**, § XII.8]. Karatzas and Shreve [**23**] offer an analogous transformation in the Brownian case. Our framework is a little different, as we have moved away from processes with independent increments to include the Bessel case with zero set defined by a stable subordinator. However, by a combination of Brownian scaling and a rearrangement of intervals in the zero set, we come up with the following:

THEOREM 1.3. *Let $(B_t, t \geq 0)$ be a Brownian motion on $\mathbb{R}$, or a Bessel process of dimension d on $\mathbb{R}^+$, for $0 < d < 2$, starting at $B_0 = 0$. Let $T_1 = \inf\{t\colon S_t = 1\}$ where $(S_t, t \geq 0)$ is the local time process of B at 0. Let $(B_t^{\#}, 0 \leq t \leq 1)$ be derived from $(B_u, 0 \leq u \leq T_1)$ by Brownian scaling:*

$$B_t^{\#} = \frac{1}{\sqrt{T_1}} B(tT_1), \quad \text{where } 0 \leq t \leq 1.$$

Let U be distributed uniformly on $[0, 1]$, independently of $(B_t^{\#})$. Let (G, D) be the maximal interval free of zeros of $B^{\#}$ which contains U, so $G < U < D$ and $B_G^{\#} = B_D^{\#} = 0$. Let $H = 1 - (D - G)$, and let Y be the process with lifetime H obtained by excising the excursion of $X^{\#}$ away from zero on the interval (G, D), and closing up the gap:

$$Y_t = \begin{cases} B_t^{\#} & \text{if } 0 \leq t \leq G, \\ B_{D+t-G}^{\#} & \text{if } G \leq t \leq H. \end{cases}$$

Then

$$(Y_t, 0 \leq t \leq H) \stackrel{d}{=} (B_t, 0 \leq t \leq G_1), \tag{1.o}$$

where $G_1 = \sup\{t\colon t \leq 1, B_t = 0\}$.

The process $(B_t^{\#}, 0 \leq t \leq 1)$ we call a *pseudo-bridge*. According to Theorem 5.2 below, which extends the result of Biane, Le Gall and Yor [**3**] from the Brownian case to the Bessel case, the law of the pseudo-bridge has density constant S_1 relative to that of the standard bridge $(B_t, 0 \leq t \leq 1 \mid B_1 = 0)$, where S_1 is the bridge local time at zero up to time 1.

We note the following description, similar to (1.h), of the process on the right-hand side of (1.o): the random time G_1 has beta$(\alpha, 1-\alpha)$ law for $\alpha = 1 - \frac{1}{2}d$ (see Dynkin [**9**]), and given G_1, $(B_t, 0 \leq t \leq G_1)$ is a Brownian or Bessel bridge of length G_1.

Theorem 1.3 contains Theorems 1.1 and 1.2 implicitly. To see this, just note that $\mathbf{V}(1)$ and $1 - G_1$ are measurable functions of $(B_t, 0 \leq t \leq G_1)$, and hence have the same joint law as the corresponding functions of $(Y_t, 0 \leq t \leq H)$, which are $\mathbf{V}(T_s)/T_s$ and $D - G$ by construction. Spelled out even further,

$$(Y_t, 0 \leq t \leq H\,;\, 1 - H\,;\, \mathbf{V}(T_1)/T_1\,;\, T_1^{-1/\alpha}) \stackrel{d}{=} (B_t, 0 \leq t \leq G_1\,;\, 1 - G_1\,;\, \mathbf{V}(1)\,;\, S_1).$$

In the Brownian case, two variables that can be added to the first list are the sign of $B^{\#}$ on (G, D) (that is, the sign of B on (GT_1, DT_1)), and $\Gamma_+(T_1)/T_1$, where

$\Gamma_+(t)$ is the time spent positive by B up to time t. Here the sign is independent of $(Y_t, 0 \leq t \leq H)$, positive with probability $\frac{1}{2}$, and by construction

$$\frac{\Gamma_+(T_1)}{T_1} = \int_0^H 1(Y_t > 0)\, dt + (1-H)1(+),$$

where $1(+)$ is the indicator of the event that $B^{\#}$ is positive on (G, D). The corresponding variables in the second list are the sign of B on $(G_1, 1)$, and $\Gamma_+(1)$, for similar reasons.

One would naturally like to extend the definition of Y beyond time H and up to time 1 to get a Brownian motion for $0 \leq t \leq 1$. This is done in Theorem 6.3 by continuing Y after time H with a signed Brownian (or Bessel) meander of length $1-H$, which given H is independent of Y on $[0, H]$. This meander could be created by a transformation of the signed excursion of length $1-H$ which was excised from $B^{\#}$ to obtain Y, or by further randomisation. But we do not see any particular way of doing this with interesting implications.

The rest of the paper is organised in sections as follows:

2. *Reformulation and proof of Theorems* 1.1 *and* 1.2. Theorem 1.1 is reformulated as Theorem 2.1. Theorem 2.1 is then proved using Lemma 2.2, which describes in particular the joint distribution of X and N^-, where N is a point process, and N^- is obtained from N by deletion of a point X picked from the points of N in an arbitrary probabilistic way.

3. *Analysis at an independent exponential time.* This section offers an alternative derivation of Theorems 1.1 and 1.2. A substitute for Lemma 2.2 is provided by Theorem 3.1. This gives a remarkable description of the joint distribution of X and N^-, where N^- is derived from a Poisson process N by removal of a single point X picked from N by h-biased sampling for some positive function h: a random variable Y is created such that conditionally given Y, X and N^- are independent, and N^- is still Poisson.

4. *Application to occupation times.* Here we recover the result of [2] concerning the joint distribution of the occupation times in sectors of Walsh's singular diffusion in the plane.

5. *Results in terms of bridges.*

6. *Rearrangements.* Here we consider the conditional distribution of the whole process given the interval partition it creates on $(0, t)$. This leads to the derivation of Theorem 1.3.

7. *More general subordinators.* Here we formulate some of the results in the stable case in a way that makes sense for more general subordinators, and we indicate some converses of Theorems 1.1 and 1.2.

2. *Reformulation and proof of Theorems* 1.1 *and* 1.2

We assume throughout this section that $(T_s, s \geq 0)$ is a stable subordinator with index α. As is well known, the Laplace transform of T_s must then be of the form

$$\text{(2.a)} \quad E[e^{-\theta T(s)}] = \exp(-cs\theta^{\alpha}) = \exp\left(-s \int_0^{\infty} (1 - e^{-\theta x})\Lambda(dx)\right), \quad \text{with } \theta > 0,$$

where $c>0$ is a scale constant, and Λ is the Lévy measure on $(0,\infty)$ defined by

$$\Lambda(dx)=\frac{c\alpha}{\Gamma(1-\alpha)}\frac{dx}{x^{\alpha+1}},\quad \text{where } 0<x<\infty. \tag{2.b}$$

Think of the range of the subordinator as a subset of a *real time* scale $t\geq 0$ parameterised by a *local time* scale $s\geq 0$. Assume $(T_s,\ s\geq 0)$ is right continuous, and let $(S(t),\ t\geq 0)$ be the continuous local time process that is the inverse of the subordinator. So

$$S(t)=\inf\{s:\ T(s)>t\};\quad T(s)=\inf\{t:\ S(t)>s\}.$$

The counting process associated with the local times and jump lengths of the subordinator is a Poisson process, homogeneous in the local time, with intensity measure Λ for jump lengths per unit local time.

We start with an elementary remark, namely that the distribution of $S(t)$ for any fixed t can be derived from that of T_s for any fixed s. Indeed, for any $t>0$ and $x>0$,

$$\begin{aligned}
P(t^{-\alpha}S(t)<x) &= P(S(t)<t^{\alpha}x)\\
&= P(T(t^{\alpha}x)>t)\quad \text{(by the inverse relation)}\\
&= P(T(t^{\alpha}x/a^{\alpha})>t/a)\quad \text{(for any } a>0\text{, by (1.i))}\\
&= P(T(s)>(s/x)^{1/\alpha})\quad \text{(by choice of } a=t(x/s)^{1/\alpha})\\
&= P(s/T_s^{\alpha}<x).
\end{aligned}$$

To summarise, for $t>0$ and $s>0$,

$$\frac{S(t)}{t^{\alpha}}\overset{d}{=}\frac{s}{T(s)^{\alpha}}. \tag{2.c}$$

In particular, we note that to get the usual normalisation of local time in the *Brownian case*, so as to make $S(t)\overset{d}{=}|B_t|$ for each fixed t, the parameters are $\alpha=\frac{1}{2}$ and $c=\sqrt{2}$. Then (2.c) gives $T(s)\overset{d}{=}s^2/B_1^2$, and hence (1.e), as noted by Lévy [**29**].

Consider now the sequence $\mathbf{V}(t)$ of ranked interval lengths as in Theorem 1.1. It is well known that

$$S(t)=\lim_{\varepsilon\to 0}\frac{\#\{i:\ V_i(t)>\varepsilon\}}{\Lambda(\varepsilon,\infty)},\quad \text{for } t>0, \tag{2.d}$$

where

$$\Lambda(\varepsilon,\infty)=\frac{c\varepsilon^{-\alpha}}{\Gamma(1-\alpha)} \tag{2.f}$$

is the Lévy measure for jump lengths greater than ε, and the convergence in (2.d) is uniform on bounded t-intervals almost surely. Formula (2.d) shows that the local time $S(t)$ may be regarded a measurable function of $\mathbf{V}(t)$, say

$$S(t)=\mathrm{Loc}[\mathbf{V}(t)]. \tag{2.g}$$

For any strictly positive, fixed or random normalizing factor X, (2.d) and (2.f) give

$$\text{Loc}\left[\frac{\mathbf{V}(t)}{X}\right] = \frac{S(t)}{X^{\alpha}} \tag{2.h}$$

simultaneously for all $t \geq 0$ a.s., and hence also for random times as well as fixed ones.

Now fix $t > 0$ and $s > 0$, and consider the assertion of Theorem 1.1, namely that

$$\frac{\mathbf{V}(t)}{t} \stackrel{d}{=} \frac{\mathbf{V}(T_s)}{T_s}. \tag{2.i}$$

Applying Loc to both sides shows that (2.i) implies (2.c), and indeed

$$\left(\frac{\mathbf{V}(t)}{t}, \frac{S(t)}{t^{\alpha}}\right) \stackrel{d}{=} \left(\frac{\mathbf{V}(T_s)}{T_s}, \frac{s}{T(s)^{\alpha}}\right). \tag{2.j}$$

This in turn is equivalent by the same kind of trick to

$$\frac{\mathbf{V}(t)}{S(t)^{1/\alpha}} \stackrel{d}{=} \frac{\mathbf{V}(T_s)}{s^{1/\alpha}}. \tag{2.k}$$

So the random scaling can be transferred from one side to the other. Since for each s the sequence $\mathbf{V}(T_s)/s^{1/\alpha}$ just consists of the descending sequence of points of a Poisson point process governed by Λ (abbreviated PPP(Λ)), and since the terms of $\mathbf{V}(t)$ are also descending, Theorem 1.1 can be reformulated as follows:

THEOREM 2.1. *For Borel subsets B of $(0, \infty)$, let*

$$N_t(B) = \#\{i\colon\ V_i(t)/S_t^{1/\alpha} \in B\}.$$

Then for each fixed $t > 0$, the counting process N_t is a PPP(Λ).

Let $X_t = A_t/S_t^{1/\alpha}$ be the point of N_t corresponding to the final interval of length A_t as in (1.m). In view of (2.g), Theorem 1.2 amounts to the assertion

(2.m) *X_t is a point of N_t picked at random by size-biased sampling:*

$$P(X_t \in dx \mid N_t) = xN_t(dx) \Big/ \int yN_t(dy).$$

To summarise the above discussion: Theorems 1.1 and 1.2 admit an equivalent formulation as Theorem 2.1 and (2.m). The rest of this section offers a proof of the latter results by application of the following general lemma. An alternative approach is offered in the following section.

LEMMA 2.2 (Poisson sampling). *Let $(\Omega^{\#}, \mathbf{F}^{\#})$ be the space of counting measures on a measurable space $(S, \mathbf{S})$. Let Q be the law on $(\Omega^{\#}, \mathbf{F}^{\#})$ of a* PPP(μ) *over $(S, \mathbf{S})$. Let $f(x, n)$ be a non-negative jointly measurable function of a point $x \in S$ and a counting measure $n \in \Omega^{\#}$. Let X be a random point in S and N^- a random counting measure over S defined jointly on some basic probability space $(\Omega, \mathbf{F}, P)$. Let $N = N^- + \delta_X$ be the point process obtained from N^- by addition of a single*

extra point at X. The following statements are equivalent:

(2.n) $$P(X \in dx, N^- \in dn) = f(x, n + \delta_x)\mu(dx)Q(dn);$$

(2.o) $$P(X \in dx, N \in dn) = f(x, n)n(dx)Q(dn).$$

Proof. Let Q_x denote the distribution of $n + \delta_x$ when n is a PPP(μ) with law Q. By a change of variable, (2.n) amounts to

(2.p) $$P(X \in dx, N \in dn) = f(x, n)\mu(dx)Q_x(dn).$$

But it is well known that $(Q_x, x \in S)$ serves as a family of *Palm distributions* for Q, meaning that there is the identity of measures on $(\Omega^\# \times S, \mathbf{F}^\# \otimes \mathbf{S})$,

(2.q) $$\mu(dx)Q_x(dn) = n(dx)Q(dn).$$

See, for instance, Daley and Vere-Jones [7, § 12.1].

We note that (2.o) displays the most general possible joint distribution of X and N where the law of N is absolutely continuous with respect to Q, and X is a point of N. The proof shows how to formulate the lemma for a more general distribution Q on the space of counting measures $(\Omega^\#, \mathbf{F}^\#)$ instead of the law of a PPP. Simply replace Q in (2.n) by the Q_x distribution of $n - \delta_x$, where $(Q_x, x \in S)$ serves as a family of Palm distributions for Q relative to μ, as in (2.q). The existence of such a disintegration of the law Q of a point process is known in great generality (see, for example, Kallenberg [**20**]).

Proof of Theorem 2.1 *and* (2.m). Let Q be the law of a PPP(Λ). Theorem 2.1 and (2.m) combined amount to

(2.r) $$P(X_t \in dx, N_t \in dn) = \frac{x}{\int yn(dy)} n(dx)Q(dn),$$

which, by Lemma 2.2, is equivalent to

(2.s) $$P(X_t \in dx, N_t^- \in dn) = \frac{x}{x + \int yn(dy)} \Lambda(dx)Q(dn),$$

where $N_t^- = N_t - \delta_{X_t}$ is the point process N_t with the point at X_t removed. Thus N_t^- has a point at $V_n(t)/S_t^{1/\alpha}$ for each complete excursion interval $V_n(t)$. Now it is clear in principle that both X_t and N_t^- are functions of the point process M on $(0, \infty)^2$ defined by the jumps of the subordinator $(T_s, s \geq 0)$:

$$M(B) = \#\{s\colon (s, T_s - T_{s-}) \in B\}, \quad \text{with } B \subset (0, \infty)^2,$$

where M is a PPP($ds\Lambda(dx)$). We proceed to verify (2.s) by a change of variables from the distribution of M. Let $Y_t = D_t - G_t$ be the length of the jump interval covering t. Given $M = m$, (S_t, Y_t) is a.s. the unique point (s, y) of M such that $T_{s-} < t$ and $T_{s-} + y \geq t$, where $T_{s-} = T_{s-}(m) = \int x1(s' < s)m(ds'\, dx)$. That is to say,

$$P(S_t \in ds, Y_t \in dy, M \in dm) = 1(T_{s-} < t, T_{s-} + y \geq t)m(ds\, dy)P(M \in dm).$$

Let M^- be M with the point (S_t, Y_t) removed. By Lemma 2.2,

(2.t) $$P(S_t \in ds, Y_t \in dy, M^- \in dm) = 1(T_{s-} < t, T_{s-} + y \geq t)\, ds\Lambda(dy)P(M \in dm),$$

where $T_{s-} = T_{s-}(m)$. Now for a fixed $s > 0$, let K_s be the point process on $(0, \infty)$ obtained by first restricting M to $(0, s) \times (0, \infty)$, then projecting onto the second coordinate and scaling via the transformation

$$(s', x) \to x/s^{1/\alpha}. \tag{2.u}$$

That is to say, for subsets C of $(0, \infty)$,

$$K_s(C) = \#\{s' : s' < s, (T_{s'} - T_{s'-})/s^{1/\alpha} \in C\}.$$

For each fixed s, the point process K_s is a PPP(Λ), with law denoted by Q. Now $T_{s-} = T_{s-}(m) = s^{1/\alpha} T(k_s)$, where k_s is the image of m under the scaling transformation (2.u), and $T(n) = \int x n(dx)$. Since N_t^- as in (2.s) is just $N_t^- = K_{S_t}$, a change of variables from (2.t) shows that

$$P(S_t \in ds, Y_t \in dy, N_t^- \in dn) = 1(s^{1/\alpha} T < t, s^{1/\alpha} T + y \geq t)\, ds \Lambda(dy) Q(dn) \tag{2.v}$$

where $T = T(n)$. Since $X_t = A_t / S_t^{1/\alpha} = t/S_t^{1/\alpha} - T(N_t^-)$, we get

$$P(S_t \in ds, Y_t \in dy, X_t \in dx, N_t^- \in dn) = 1\left(\frac{t}{s^{1/\alpha}} - T \in dx, x \leq \frac{y}{s^{1/\alpha}}\right) ds \Lambda(dy) Q(dn),$$

$$\begin{aligned} \frac{P(X_t \in dx, N_t^- \in dn)}{Q(dn)} &= \int\int ds \Lambda(dy) 1\left(\frac{t}{s^{1/\alpha}} - T \in dx, x \leq \frac{y}{s^{1/\alpha}}\right) \\ &= \int ds 1\left(\frac{t}{s^{1/\alpha}} - T \in dx\right) \Lambda(s^{1/\alpha} x, \infty) \\ &= |dx| \left|\frac{ds}{dx}\right| \Lambda(s^{1/\alpha} x, \infty), \end{aligned}$$

where

$$\frac{t}{s^{1/\alpha}} - T = x, \quad s = \left(\frac{t}{x + T}\right)^{\alpha}, \quad s^{1/\alpha} x = \frac{tx}{x + T}, \quad \frac{ds}{dx} = -\alpha \frac{t^{\alpha}}{(x + T)^{1+\alpha}}.$$

Thus

$$\begin{aligned} \frac{P(X_t \in dx, N_t^- \in dn)}{Q(dn)} &= |dx| \, \alpha \frac{t^{\alpha}}{(x + T)^{\alpha+1}} \frac{c}{\Gamma(1 - \alpha)} \left(\frac{xt}{x + T}\right)^{-\alpha} \\ &= \Lambda(dx) \frac{x}{x + T} \quad \text{by (2.b).} \end{aligned}$$

3. *Analysis at an independent exponential time*

This section presents an alternative approach to Theorems 1.1 and 1.2, based on analysis of the intervals up to an independent exponential time. As in § 2, we establish Theorems 1.1 and 1.2 as reformulated in Theorem 2.1 and (2.m). The argument brings out some remarkable general features of size-biased sampling from a Poisson point process, which are presented in Theorem 3.1.

Let T be an exponentially distributed random variable with rate 1 independent of the stable subordinator (T_s). Let N_T be the point process whose points in descending order form the normalised sequence of ranked interval lengths

$\mathbf{V}_T/S_T^{1/\alpha}$. Let $X_T = A_T/S_T^{1/\alpha}$, which is one of the points of N_T. Since it is clear *a priori* by scaling invariance that N_T is independent of T, it suffices to show that

(3.a) N_T is a PPP(Λ);

(3.b) *given T and N_T, the point X_T is picked from N_T by size-biased sampling.*

For the rest of the argument, we write simply S for S_T, and ν for $1/\alpha$. Each length in the sequence $\mathbf{V}_T$ represents either the length of one of the infinite number of jumps of the subordinator occurring at some local time strictly before S, or the age A_T of the excursion in progress at time T. The counting process N_T associated with $\mathbf{V}_T/S^\nu$ is therefore the sum of two processes:

(3.c) *the process N_T^- counting the lengths of jumps of the subordinator strictly before local time S, with each jump normalised by S^ν;*

(3.d) *the process counting a single point at $X_T = A_T/S^\nu$.*

It will now be argued that these two counting processes are conditionally independent given S, with just the right conditional distributions given S to make (3.a) and (3.b) hold.

Think of T as the time of the first mark in a Poisson process of marks with rate 1 on the real time scale, independent of the subordinator. Associate with each jump of the subordinator the real times of any marks that fall in that jump interval, measured from the start of the jump (i.e. the beginning of the corresponding excursion). A big Poisson process, homogeneous in the local time coordinate, is then obtained by counting the local times of jumps of the subordinator, jointly with jump lengths embellished by their mark times if any. And S is the local time of the first marked jump in this big Poisson process. This idea has been used in a number of contexts. See, for instance, Greenwood and Pitman [**15**], Rogers [**38**, **39**]. Rogers and Williams [**40**, § VI.53] give a careful formulation. The independent decomposition of this Poisson point process into points with marks and points without implies the following formulae, where we assume for simplicity (and without loss of generality) that the constant c in (2.a) is $c = 1$:

(3.f) *S is exponential with rate $\int (1 - e^{-x})\Lambda(dx) = 1$;*

(3.g) *the local times and lengths of the unmarked jumps appear according to a homogeneous Poisson point process with intensity measure $e^{-x}\Lambda(dx)$ per unit local time, independently of S;*

(3.h) *independently of S and all unmarked jumps, the age A_T, which is the time of the first mark measured from the beginning in the first marked jump interval, has the* gamma$(1-\alpha, 1)$ *distribution:*

$$P(A_T \in dx) = e^{-x}\, dx \Lambda(x, \infty) = \frac{dx}{\Gamma(1-\alpha)} e^{-x} x^{-\alpha}, \quad \text{with } x > 0.$$

Now condition on $S = s$, and normalise all the lengths by s^ν, where $\nu = 1/\alpha$. From (3.g) by the linear change of variable $x = y/s^\nu$, the process of normalised completed jumps is Poisson with mean measure

(3.i) $$s^\nu \exp(-s^\nu x) \frac{\alpha}{\Gamma(1-\alpha)} \frac{s\, dx}{(xs^\nu)^{\alpha+1}} = \exp(-s^\nu x)\Lambda(dx), \quad \text{where } x > 0.$$

And independently of this Poisson counting process given $S = s$, the normalised age $X_T = A_T/S^\nu$ has gamma$(1-\alpha, s^\nu)$ distribution: for $x > 0$,

$$P(X_T \in dx \mid S = s) = \frac{dx}{\Gamma(1-\alpha)} \exp(-s^\nu x) x^{-\alpha} (s^\nu)^{1-\alpha} \tag{3.j}$$
$$= \nu s^{\nu-1} x \exp(-s^\nu x) \Lambda(dx).$$

Now (2.a), (3.f), (3.i) and (3.j) show that conditions (d), (e) and (f) of the next theorem are satisfied with

$$N = N_T, \quad Y = S^\nu, \quad X = X_T, \quad h(x) = x, \quad \mu = \Lambda.$$

The desired conclusions (3.a) and (3.b) then follow at once from conditions (a) and (c) of the next theorem.

THEOREM 3.1. *Let $(\Omega^\#, \mathbf{F}^\#)$ be the space of counting measures on a measurable space $(S, \mathbf{S})$, let μ be a σ-finite measure on S. Let h be a non-negative measurable function defined on S such that*

$$\text{if } N \text{ is a PPP}(\mu) \text{ then } P(0 < Nh < \infty) = 1, \tag{3.k}$$

where $Nh = \int h\, dN$. Suppose Y is a non-negative random variable, X a random point in S, and N^- a point process on S, all defined on some basic probability space $(\Omega, \mathbf{F}, P)$. Let $N = N^- + \delta_X$ be the point process obtained by the addition to N^- of a single extra point at X. The following statements (a), (b) *and* (c) *combined are equivalent to* (d), (e) *and* (f) *combined:*

(a) *N is a* PPP(μ),
(b) *$Y = T/Nh$ where T has standard exponential distribution independently of N,*
(c) *given N and Y, X is picked from N by h-biased sampling;*

(d) $$P(Y > y) = \exp\Big[-\int (1 - e^{-h(x)y}) \mu(dx) \Big],$$

(e) *given $Y = y$, X and N^- are conditionally independent, with*
$$P(X \in dx \mid Y = y) = q(y)^{-1} h(x) e^{-h(x)y} \mu(dx)$$
for a constant of integration $q(y)$,

(f) *given $Y = y$, N^- is a* PPP$[e^{-h(x)y}\mu(dx)]$.

Proof. Since the joint law of either of the triples (N, X, Y) or (N^-, X, Y) determines that of the other, it suffices to establish that (d), (e) and (f) hold for any particular triple (N, X, Y) satisfying (a), (b) and (c). This is conveniently done by supposing that N is of the form

$$N(B) = \sum_i 1(X_i \in B)$$

for some sequence of S-valued random variables (X_i), and that given (X_i) there is a sequence of independent variables (Y_i) such that Y_i is exponential with rate $h(X_i)$. Let $Y = \min_i Y_i = Y_I$ say, and let $X = X_I$. Then (a) holds by construction, and (b) and (c) follow from standard properties of independent exponential variables. It is well known that if the points of a Poisson process on S are assigned

marks independently according to a transition probability function from S to some other space, the result is a Poisson process on the product space. Thus the process N^* on $S \times (0, \infty)$ defined by

$$N^*(\cdot) = \sum_i 1[(X_i, Y_i) \in \cdot]$$

is a PPP$[\mu(dx)q(x, y)\,dy]$, where $q(x, y) = h(x)e^{-h(x)y}$ is the conditional density of Y_i given $X_i = x$. Now (X_I, Y_I) is picked from N^* by the deterministic rule of taking the point with minimum y-value, and N^- is the projection of $N^* - \delta_{(X_I, Y_I)}$ onto the x-axis. So (d), (e) and (f) follow immediately from standard Poisson features of N^*.

We continue this section with a series of remarks concerning Theorem 3.1.

REMARK 3.2. Assuming that (a) holds, we can recognise the right-hand side of (e) as the Laplace transform of Nh:

$$P(Y > y) = E \exp[-yNh] = \exp[-F(y)], \tag{3.m}$$

say. Then the constant of integration $q(y)$ in (d) turns out to be simply the derivative of F at y:

$$q(y) = F'(y). \tag{3.n}$$

REMARK 3.3. Hidden in Theorem 3.1 is the following consequence of Lemma 2.2, noted already in a special case in (2.s): *if N^- is obtained by deleting a point X from N, a* PPP(μ), *by h-biased sampling, then*

$$P(X \in dx, N^- \in dn) = \frac{h(x)}{nh + h(x)}\,\mu(dx)Q(dn), \tag{3.p}$$

where Q is the P distribution of N. Suppose Y is constructed as in condition (b) of the theorem, so $Y = T/Nh$ for a standard exponential variable T independent of N and X. Then (a), (b) and (c) hold, and it is interesting to see how (3.p) emerges from (d), (e) and (f). Let Q_y denote the law of a PPP(μ_y) where $\mu_y(dx) = e^{-h(x)y}\mu(dx)$. So Q_y is the law of N^- given $Y = y$ as prescribed in (f). It is easy to check that

$$Q_y(dn) = \frac{e^{-(nh)y}}{\phi(y)}\,Q(dn)$$

where $nh = \int n(dx)h(x)$, and

$$\phi(y) := \int Q(dn)e^{-(nh)y} = P(Y > y)$$

by (3.m). Now (3.m) and (3.n) make

$$P(Y \in dy) = -\phi'(y)\,dy = \phi(y)q(y)\,dy.$$

So conditions (d), (e) and (f) of Theorem 3.1 imply that

$$P(X \in dx, N^- \in dn) = \int P(X \in dx, N^- \in dn \mid Y = y)P(Y \in dy)$$

$$= \left[\int q(y)^{-1}h(x)e^{-h(x)y}\frac{e^{-(nh)y}}{\phi(y)}\phi(y)q(y)\,dy\right]Q(dn)\mu(dx),$$

which reduces to (3.p).

REMARK 3.4. Suppose again that N is a PPP(μ) and X is picked from N by h-biased sampling. According to Theorem 3.1, provided that the space $(\Omega, \mathbf{F}, P)$ on which N and X are defined allows room for a continuously distributed random variable T independent of both N and X, a random variable Y can be created from N and T such that conditionally on Y, N^- is a Poisson process independent of X. In particular, this implies that the law of N^- is a mixture of Poisson laws, a fact which does not seem obvious *a priori*, and is even well hidden in formula (3.p). In common terminology, N^- is a *Cox process*.

REMARK 3.5. The part of the assumption in (3.k) that $Nh > 0$ a.s. rules out the case when μ is a finite measure. But this assumption was made only for simplicity of exposition. It is clear from the proof that the result still holds without assuming $Nh > 0$ a.s. provided the various conditions are modified to allow the possibility of the event $(Y = \infty)$, in which case X is undefined and N^- is identically zero. In particular, in the case where S is a single point, μ and h are identified with positive real numbers, and N with a non-negative integer-valued random variable, Theorem 3.1 reduces to the following elementary fact: N *has* Poisson(μ) *distribution on* $\{0, 1, 2, \ldots\}$ *if and only if the distribution of* $(N-1)$ *restricted to the event* $N \geq 1$ *is the mixture over* λ *of* Poisson(λ) *distributions with respect to* $d\lambda e^{\mu-\lambda}1(\lambda \leq \mu)$. (This fact is obvious by conditioning on the time of the last arrival before time μ in a Poisson process with rate 1 on $(0, \infty)$.)

REMARK 3.6. Say that (U, V, W) is Markov if U and W are conditionally independent given V. A curious feature of the joint law described in Theorem 3.1 is that (Y, N, X) is Markov, while $(N - \delta_X, Y, X)$ is also Markov. A third equivalent description of the Markov triple (Y, N, X) is obtained by combining (d) (the law of Y) and (c) (the law of X given Y and N) with the following formula for the law of N given Y:

$$P(N \in dn \mid Y = y) = c(y)^{-1}(nh)e^{-y(nh)}Q(dn), \tag{3.q}$$

where Q is the law of a PPP(μ), and $c(y)$ is a constant of integration which can be calculated variously as

$$c(y) = \int (nh)e^{-y(nh)}Q(dn) = -\phi'(y) = \phi(y)q(y) = P(Y \in dy)/dy.$$

This is an easy consequence of (e) and the following proposition, which is applied conditionally given $Y = y$, with $e^{-h(x)y}\mu(dx)$ instead of $\mu(dx)$, and $h(x)/q(y)$ instead of $h(x)$.

PROPOSITION 3.7. *Suppose under P that N^- is a* PPP(μ) *with distribution Q, and $N = N^- + \delta_X$ where X is independent of N^- with law*

$$P(X \in dx) = h(x)\mu(dx)$$

for some density h. Then N has law

$$P(N \in dn) = (nh)Q(dn),$$

and X is picked from N by h-biased sampling.

Proof. Apply the Poisson sampling Lemma 2.2.

To conclude this section, we mention two natural extensions of Theorem 3.1 which can be obtained by minor variations of the same argument. One is a description of the joint distribution of N and $m \geq 1$ points of N picked by repeated h-biased sampling, which is given by Perman, Pitman and Yor [**35**]. The other, spelled out as Theorem 3.8 below, is obtained by using a more general transition density $q(x, y)$, with $x \in S$, $y \geq 0$, instead of $q(x, y) = h(x)e^{-h(x)y}$. We do not know of any interesting applications of this result, but it seems nonetheless worth recording:

THEOREM 3.8. *Let $q(x, y)$, with $x \in S$ and $y \geq 0$, be a transition probability density from S to $(0, \infty)$. Let*

$$F(x, y) = \int_0^y q(x, u)\, du,$$

$$G(x, y) = 1 - F(x, y) = \int_y^\infty q(x, u)\, du.$$

For $y > 0$ define

$$q(y) = \int q(x, y)\mu(dx),$$

$$F(y) = \int_0^y q(u)\, du = \int F(x, y)\mu(dx).$$

Assume $F(y) < \infty$ for all $y > 0$. Suppose there is defined on $(\Omega, \mathbf{F}, P)$ a triple (N, X, Y) such that

(a) N *is a* PPP(μ),

(b) $$P(Y > y \mid N) = \prod_i G(X_i, y),$$

(c) $$P(X \in dx \mid N, Y = y) = \frac{h(x, y)N(dx)}{\int h(x, y)N(dx)},$$

where $h(x, y) = q(x, y)/G(x, y)$ is the hazard rate at y associated with the density $q(x, \cdot)$. Let $N^- = N - \delta_X$. Then

(d) $$P(Y > y) = \exp(-F(y)),$$

(e) *given* $Y = y$, X *and* N^- *are conditionally independent, with*

$$P(X \in dx \mid Y = y) = \frac{q(x, y)}{q(y)} \mu(dx),$$

(f) *given* $Y = y$, N^- *is a* PPP $G(x, y)\mu(dx)$.

Conversely, if (N^-, X, Y) *satisfy* (d), (e) *and* (f), *and* $N = N^- + \delta_X$, *then* (a), (b), (c) *hold for* (N, X, Y).

Notes. (i) Taking expectations in (b) gives

$$P(Y > y) = E \exp - \int \log G(x, y) N(dx) \tag{d'}$$

which can be used as a substitute for (d).

(ii) Integrating out y gives the formula

$$P(X \in dx, N^- \in dn) = \mu(dx) P(dn) \int q(x, y) \exp\left[\int \log G(x', y) n(dx') \right] dy.$$

4. *Application to occupation times*

Barlow, Pitman and Yor [**2**] found the joint distribution of the occupation times in sectors of Walsh's singular diffusion in the plane. This result, reformulated here slightly more generally as Theorem 4.1, is an extremely close relative of Theorem 1.1. Indeed, it was Theorem 4.1 that first led us to Theorem 1.1, and either result can easily be derived from the other.

Walsh [**45**] defined a diffusion process X in the plane as follows. Assume X starts at the origin, that the radial part of X is a reflecting Brownian motion on $[0, \infty)$, and that each excursion of the radial part away from zero corresponds to an excursion of X within a ray emanating from the origin at an angle chosen independently according to some given distribution F on $[0, 2\pi)$. See [**1**] for a more careful description of this process. In particular, in the case where F puts probability p on angle 0 and probability $1 - p$ on angle π, the process X can be identified as a *skew Brownian motion* such that

$$P(X_t > 0) = p, \quad P(X_t < 0) = 1 - p. \tag{4.a}$$

Skew Brownian motion, introduced by Itô and McKean [**18**], has been studied by Walsh [**45**], Harrison and Shepp [**16**], and Brooks and Chacon [**6**]. It appears naturally in certain limit theorems for real diffusions considered by Rosenkrantz [**41**], Le Gall [**28**], and Franchi [**13**]. And with Barlow in [**2**] we considered the process X in the plane defined as above, but whose radial part is a Bessel process of dimension $d \in (0, 2)$ instead of a reflecting Brownian motion. As pointed out by Molchanov and Ostrowski [**31**] (see also [**2**]), the inverse local time at zero of this Bessel process is a stable subordinator with index $\alpha = 1 - \frac{1}{2}d$.

Walsh's construction involves a random labelling of the jump intervals of the subordinator, which we can describe as follows. Suppose that every jump interval $I = (T_{s-}, T_s)$ of the subordinator (T_s) is labelled by a random variable Θ_I. In Walsh's construction, the Θ_I are understood to represent angles, but any measurable space of labels is now allowed. Assume these Θ_I are mutually independent as I varies, independent also of the lengths of all intervals, and that

the Θ_I have some common distribution F. To be pedantic, this means the following: for some exhaustive listing $(I(n), n = 1, 2, \ldots)$ of the maximal open intervals of Z^c, where the definition of the endpoints of $I(n)$ may depend measurably on the subordinator (T_s) whose closed range is Z, random variables $\Theta_{I(n)}$ are independent and identically distributed according to F, independently also of (T_s). This defines Θ_I for every jump interval I. And for any other such (T_s)-dependent listing of the intervals, exhaustive or not, the sequence of Θ values will have the same probabilistic properties.

Define a process $(\Theta_t, t \in Z^c)$ by

$$\Theta_t = \Theta_I \quad \text{if } t \in I,$$

and consider for a fixed or random time T the random occupation measure $\Gamma(\cdot, T)$ defined on measurable subsets D of Θ values by

$$\Gamma(D, T) = \int_0^T 1_D(\Theta_s)\, ds. \tag{4.b}$$

Put another way,

$$\Gamma(D, T) = \sum_n V_n(T) 1[\Theta_n(T) \in D], \tag{4.c}$$

where $\Theta_n(T)$ is the label of the nth longest subinterval of $Z^c \cap (0, T)$, whose length is $V_n(T)$. Provided T depends only on the subordinator, these labels are independent with common distribution F.

THEOREM 4.1 [2]. *For disjoint measurable sets of labels D_i, with $i = 1, \ldots, n$, and $s, t > 0$, the random variables*

$$\frac{\Gamma(D_i, t)}{S(t)^{1/\alpha}}, \quad \textit{for } i = 1, \ldots, n, \tag{4.d}$$

are independent with the same joint distribution as

$$\frac{\Gamma(D_i, T_s)}{s^{1/\alpha}}, \quad \textit{for } i = 1, \ldots, n. \tag{4.e}$$

Here $\Gamma(D_i, T_s)/s^{1/\alpha}$ has the same stable distribution with index α as $T(F(D_i))$. This, and the independence of $\Gamma(D_i, T_s)$ as i varies, are immediate consequences of the Poisson character of the counting process on $[0, \infty) \times$ labels which counts the local times and labels associated with jump intervals of the subordinator. As an immediate consequence of Theorem 4.1, we have the following corollary:

COROLLARY 4.2 [2]. *The random variables*

$$\frac{\Gamma(D_i, t)}{t}, \quad \textit{for } i = 1, \ldots, n; \quad \frac{S(t)}{t^{1/\alpha}} \tag{4.f}$$

have the same joint distribution as

$$\frac{\Gamma(D_i, T_s)}{T_s} \quad \textit{for } i = 1, \ldots, n; \quad \frac{s}{T_s^{1/\alpha}}. \tag{4.g}$$

Further description of the common joint law of the variables in (4.f) and (4.g) can be found in [2], along with corresponding results for the bridge obtained by conditioning X to return to zero at time t. In particular, according to formula (4.a′) of [2], for $F(D)=p$, the Stieltjes transform of the one-dimensional distribution of $\Gamma(D, 1)$ for fixed D is given for $u>0$ by

$$E\left[\frac{1}{u+\Gamma(D, 1)}\right]=\frac{p(u+1)^{\alpha-1}+qu^{\alpha-1}}{p(u+1)^{\alpha}+qu^{\alpha}}. \tag{4.h}$$

Lamperti [**27**, formula (3.17)] found the same transform for the limiting distribution of occupation times of certain discrete time processes. Lamperti inverted the transform to give the following explicit formula for the corresponding probability density, now identified as the density of $\Gamma(D, 1)$:

$$\frac{P(\Gamma(D, 1)\in dx)}{dx}=\frac{a\sin\pi\alpha}{\pi}\frac{x^{\alpha}\bar{x}^{\alpha-1}+x^{\alpha-1}\bar{x}^{\alpha}}{a^2x^{2\alpha}+2ax^{\alpha}\bar{x}^{\alpha}\cos\pi\alpha+\bar{x}^{2\alpha}}, \tag{4.i}$$

where $\bar{x}=1-x$ and $a=(1-p)/p$. In particular, for $\alpha=\frac{1}{2}$ this is the density found by Keilson and Wellner [**24**] for the fraction of time spent positive by a skew Brownian motion which is positive at each fixed time with probability p.

Derivation of Theorem 4.1 *from Theorem* 2.1. When assigned independent labels according to distribution F, the points of the Poisson process N_t in Theorem 2.1 that are assigned labels in D_i form independent Poisson processes with mean measures $F(D_i)\Lambda$, for $i=1, \ldots, n$. Since adding the values of points with labels D_i gives $\Gamma(D_i, t)/S(t)^{1/\alpha}$, these must be independent stable(α) random variables, as asserted by Theorem 4.1.

As Theorem 4.1 was proved in [2] by a different method (computation of the joint Laplace transform via excursion theory), we offer also the following argument:

Derivation of Theorem 2.1 *from Theorem* 4.1. Take the distribution of labels to be uniform on $[0, 1]$. Fix $t>0$. Theorem 4.1 implies that

$$\left(\frac{\Gamma([0, a], t)}{S(t)^{1/\alpha}}, 0\leq a\leq 1\right) \tag{4.j}$$

is a process with stationary independent increments, identical in law to $(T_a, 0\leq a\leq 1)$. So the counting process M_t defined on subsets B of $(0, \infty)$ by

$$M_t(B)=\#\left\{a: \frac{\Gamma(\{a\}, t)}{S(t)^{1/\alpha}}\in B\right\} \tag{4.k}$$

is Poisson with mean measure Λ. This process counts the sizes of the atoms of the random measure $\Gamma(\cdot, t)/S(t)^{1/\alpha}$, which for each n puts an atom of size $V_n(t)/S(t)^{1/\alpha}$ at a label chosen uniformly at random from $[0, 1]$, independently as n varies. Since there is probability zero that any two labels are equal, the counting processes N_t in Theorem 2.1 and M_t in (4.k) are identical almost surely.

In view of the argument preceding Theorem 2.1, it is now plain that Theorems 1.1, 2.1 and 4.1 are equivalent in the sense that any one of these results can easily

be derived from any of the others. But to build up any of these results from scratch seems to require some non-trivial calculation.

This section would not be complete without mentioning the analogue in this setting of Theorem 1.2. Recall from (4.c) that $\Theta_n(t)$ is the label of the nth longest subinterval of $Z^c \cap (0, t)$, whose length is $V_n(t)$.

THEOREM 4.3. *Let $\mathbf{G}_t$ be the σ-field generated by the random variables $\Theta_n(t)$ and $V_n(t)$ for $n = 1, 2, \ldots$. Then*

$$P(\Theta_t \in \cdot \mid \mathbf{G}_t) = \Gamma(\cdot, t)/t. \tag{4.l}$$

Proof. Due to the independence of labels and interval lengths, Theorem 1.2 implies that given $\Theta_n(t)$ and $V_n(t)$ for $n = 1, 2, \ldots, \Theta_t = \Theta_n(t)$ with probability $V_n(t)$. So the conclusion is immediate from formula (4.c).

On the way to his arcsine law (1.a) for Brownian motion, Lévy [**29**, formula (51)], found that

$$P(t^{-1}\Gamma_+(t) \in du \mid B_t > 0) = \frac{2}{\pi}\sqrt{\left(\frac{u}{1-u}\right)}\,du, \quad \text{for } 0 < u < 1. \tag{4.m}$$

In combination with the arcsine law (1.a), this amounts to

$$P(B_t > 0 \mid t^{-1}\Gamma_+(t) = u) = u, \quad \text{for } 0 < u < 1, \tag{4.n}$$

which is a special case of Theorem 4.3. According to Theorem 4.3, formula (4.n) must hold also in the case of a skew Brownian motion B, regardless of what probability p there is for positive excursions, and even for a 'skew Bessel process'.

5. *Results in terms of bridges*

Let P_{0t} be the law of a *bridge of length t*, obtained as a conditional distribution given $t \in Z$, where Z is the closed range of a subordinator $(T_s, s \geq 0)$. If we assume for simplicity that Z is the zero set of a Brownian motion or Bessel process $(B_t, t \geq 0)$ set up on the canonical path space, then

$$P_{0t}(\cdot) = P(\cdot \mid B_t = 0)$$

governs a Brownian or Bessel bridge of length t. And there are nice versions of conditional distributions so the basic *switching identity*

$$P(\cdot \mid T_s = t) = P_{0t}(\cdot \mid S_t = s) \tag{5.a}$$

holds exactly for every $s > 0$ and $t > 0$. In the Brownian case this is the basis of Lévy's argument (1.f). See also Karatzas and Shreve [**22**, § 4], Kallenberg [**21**, Lemma 4.1]. The switching identity allows us to reformulate Theorem 1.1 in terms of bridges:

THEOREM 5.1. *For a stable subordinator (T_s), the conditional distribution of $\mathbf{V}(t)$ given S_t is the same for the bridge distribution P_{0t} as for the original distribution P.*

Proof. Compute as follows:

$$P_{0t}\left(\frac{\mathbf{V}(t)}{t}\in\cdot\;\middle|\;S_t=s\right)=P\left(\frac{\mathbf{V}(T_s)}{T_s}\in\cdot\;\middle|\;T_s=t\right)\quad\text{(by the switching identity (5.a))}$$

$$=P\left(\frac{\mathbf{V}(T_s)}{T_s}\in\cdot\;\middle|\;\frac{s}{T_s^{\alpha}}=\frac{s}{t^{\alpha}}\right)$$

$$=P\left(\frac{\mathbf{V}(t)}{t}\in\cdot\;\middle|\;\frac{S_t}{t^{\alpha}}=\frac{s}{t^{\alpha}}\right)\quad\text{(by (2.j))}$$

$$=P\left(\frac{\mathbf{V}(t)}{t}\in\cdot\;\middle|\;S_t=s\right).$$

As we give a different approach to an extension of Theorem 5.1 in § 7, we record the following argument, which shows that Theorem 5.1 joins Theorems 1.1, 2.1 and 4.1 in a circle of 'equivalent' results, any one of which can be derived quickly from any of the others.

Derivation of Theorem 1.1 *from Theorem* 5.1. Assuming the conclusion of Theorem 5.1, we see that the quantities in the first and last lines of the preceding proof must be equal; hence so too must the quantities in the second and third lines. Take $s/t^{\alpha}=x$, say, and use scaling to deduce that

$$P\left(\frac{\mathbf{V}(T_s)}{T_s}\in\cdot\;\middle|\;\frac{s}{T_s^{\alpha}}=x\right)=P(\mathbf{V}(1)\in\cdot\;|\;S_1=x);$$

but as remarked in (2.c), $s/T_s^{\alpha}\overset{d}{=}S_1$, whence the conclusion of Theorem 1.1 for $t=1$ follows, and then that for all t by scaling.

The present discussion is closely related to the work of Biane, Le Gall and Yor [3], who considered the *pseudo-bridge*

$$B_u^{\#}=\frac{1}{\sqrt{T_1}}B(uT_1),\quad\text{with } 0\leqslant u\leqslant 1, \tag{5.b}$$

in the case where B is Brownian motion and (T_s) its inverse local time at zero. Here is a straightforward generalisation of their result:

THEOREM 5.2 (obtained by [3] in the Brownian case $\delta=1$). *Let (T_s) be the subordinator inverse of the local time at zero (S_t) of (B_t), a Brownian motion, or a Bessel process of dimension δ, for $0<\delta<2$. Then the law P_{01} of the corresponding bridge of length* 1 *and the law $P_{01}^{\#}$ of the pseudo-bridge* (5.b) *are mutually absolutely continuous with*

$$\frac{dP_{01}}{dP_{01}^{\#}}=c\Gamma(\alpha+1)S_1, \tag{5.c}$$

where $\alpha=1-\frac{1}{2}\delta$ and c is the scale constant determined by the normalisation of the local time, as in (2.a) *and* (2.b).

Proof. According to Molchanov and Ostrovski [**31**] (or see [**2**]), the subordinator (T_s) derived from a Bessel process of dimension δ is stable with index $\alpha = 1 - \frac{1}{2}\delta$. And the computation of [**3**] in the Brownian case $\alpha = \frac{1}{2}$ extends easily to a general $0 < \alpha < 1$.

THEOREM 5.3. *In the setting of Theorems* 5.1 *and* 5.2, *let* $\mathbf{G}_t$ *be the* σ*-field generated by* $\mathbf{V}(t)$. *Then the bridge law* P_{0t} *and the unconditional law* P *are mutually absolutely continuous on* $\mathbf{G}_t$, *with*

$$\frac{dP_{0t}}{dP}\Big|\mathbf{G}_t = c\Gamma(\alpha+1)\frac{S_t}{t^\alpha}. \tag{5.d}$$

Proof. By scaling, it suffices to consider the case where $t = 1$. By construction of the pseudo-bridge, $\mathbf{V}(T_1)/T_1$ is the vector of interval lengths generated by the pseudo-bridge $B^{\#}$. The identity $\mathbf{V}(T_1)/T_1 \overset{d}{=} \mathbf{V}(1)$ of Theorem 1.1 therefore amounts to

$$P_{01}^{\#} = P \quad \text{on } \mathbf{G}_1, \tag{5.e}$$

so the result follows immediately from Theorem 5.2.

It follows at once from Theorem 5.3 and the factorisation criterion for sufficient statistics that P_{0t} and P must have the same conditional distribution for $\mathbf{V}(t)$ given S_t. Thus Theorem 5.3 provides a strengthening of Theorem 5.1. Equally, Theorem 5.3 can be derived from Theorem 5.1 without reference to the pseudo-bridge by checking that the formula $S_t/E(S_t)$ gives the correct one-dimensional density on the σ-field generated by S_t. For a way to do this, see formula (7.b).

In the setting of § 4, where intervals are assigned independent and identically distributed labels, it is immediate from the independence of the subordinator and the labelling process that Theorem 5.3 still holds with $\mathbf{G}_t$ enlarged to include the labels generated up to time t, as in Theorem 4.3. And Theorem 5.1 can be similarly extended. In the Brownian case, this shows, for example, that, for $\Gamma_+(t)$ the time spent positive up to t,

$$P(\Gamma_+(t) \in \cdot \mid S_t = s,\ B_t = 0) = P(\Gamma_+(t) \in \cdot \mid S_t = s). \tag{5.f}$$

Lévy hints at this identity in the paragraph mentioned below (1.b), and he could have computed either conditional distribution by his methods. Explicit formulae for joint densities of occupation and local times of Brownian motion can be found in work by Perkins [**33**, Theorem 10] and Karatzas and Shreve [**22**, § 6.3].

6. *Rearrangements*

Our concern in this section is how to describe precisely the relation between what happens up to fixed times t and up to inverse local times T_s if we keep track of the whole zero set, or equivalently, the order in which the intervals in $\mathbf{V}(t)$ or $\mathbf{V}(T_s)$ appear as they are laid down by the process. The whole discussion can be conducted for a Bessel process, or just for a stable subordinator. But to be definite we shall assume for simplicity of exposition that we are dealing with the Brownian case $\alpha = \frac{1}{2}$. We assume the reader has some familiarity with the theory

of Brownian excursions. For background and further references, see, for example, Revuz and Yor [**37**, Chapter 12] or Rogers [**39**].

Consider various laws on $C[0, 1]$ with coordinates $(X_t, 0 \leq t \leq 1)$ and S_1 the Brownian local time at 0 up to time 1:

$$Q = \text{law of standard Brownian motion started at 0;}$$

$$Q^s = Q(\cdot \mid S_1 = s);$$

$$Q_0 = \text{law of standard Brownian bridge} = Q(\cdot \mid X_1 = 0);$$

$$Q_0^s = Q_0(\cdot \mid S_1 = s) = Q(\cdot \mid X_1 = 0, S_1 = s).$$

Also, for a distribution μ on $(0, \infty)$, let

$$Q^\mu = \int Q^s \mu(ds),$$

and define Q_0^μ similarly in terms of the Q_0^s. Let

$$\nu(ds) = \frac{2}{\sqrt{(2\pi)}} e^{-s^2/2}\, ds \quad \text{(the } Q \text{ distribution of } S_1),$$

$$\nu_0(ds) = s e^{-s^2/2}\, ds \quad \text{(the } Q_0 \text{ distribution of } S_1).$$

Then $Q^\nu = Q$, $Q_0^{\nu_0} = Q_0$. According to Theorem 5.2, Q_0^ν is the law of the pseudo-bridge (5.b).

On the path space $C[0, 1]$, define $\mathbf{V} = (V_1, V_2, \ldots)$ to be the vector of interval lengths. Under any of the above laws,

$$V_1 > V_2 > \ldots \quad \text{and} \quad \sum_n V_n = 1 \text{ a.s.}$$

Then we have the following basic result:

PROPOSITION 6.1. *For every probability distribution μ on $(0, \infty)$, the distribution of $\mathbf{V}$ is the same under Q^μ as under Q_0^μ.*

Proof. The result for $\mu = \delta_s$ is Theorem 5.1, and the result for general μ follows by mixing on s.

Let (L_n, R_n) be the interval which contributes length $V_n = R_n - L_n$. So

$$\{t\colon 0 < t < 1, X_t \neq 0\} = \bigcup_n (L_n, R_n). \tag{6.a}$$

The stochastic structure of the zero set of X is thus defined by the joint distribution of any two of the random sequences $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{V}$. It seems most convenient to consider $\mathbf{V}$ and $\mathbf{R}$. Inasmuch as the law of $\mathbf{V}$ is the same under both Q^μ and Q_0^μ, whereas the structure of the whole zero set is obviously different for Q^μ and Q_0^μ, the difference lies in the conditional distribution of $\mathbf{R}$ given $\mathbf{V}$.

The law of $\mathbf{R}$ given $\mathbf{V}$ is simplest to describe under Q_0^μ, and is the same regardless of μ, the distribution of local time. Informally, for any n, the intervals of length $V_1, \ldots, V_n$ are equally likely to appear in any of $n!$ possible orderings. These orderings have an obvious consistency property as n varies. And once these orderings are determined, the value of R_n can be calculated as the sum of V_i over

all i such that V_i is put before V_n. This can be expressed neatly by introducing the random variables on $C[0, 1]$,

$$U_n = S(R_n)/S(1) \tag{6.b}$$

where $(S(t), t \geq 0)$ is the local time process. Here $U_n < U_m$ means that the interval of length V_n is placed to the left of the interval of length V_m. Equivalently: $R_n < R_m$. And

$$R_m = \sum_n V_n 1(U_n \leq U_m), \quad \text{for } m = 1, 2, \ldots . \tag{6.c}$$

This formula makes sense and holds both Q a.s. and Q_0^μ a.s. for any μ. In view of this formula, to specify the joint law of $\mathbf{V}$ and $\mathbf{R}$ under any of these probabilities, it suffices to specify the joint law of $\mathbf{V}$ and $\mathbf{U} = (U_1, U_2, \ldots)$.

PROPOSITION 6.2. *Under Q_0^μ, for any μ, the random variables $U_1, U_2, \ldots$ are independent with uniform* $[0, 1]$ *distribution, and are independent of* $\mathbf{V}$.

Proof. First, it is enough to consider the case where $\mu = \delta_s$ for $0 < s < \infty$. But since $Q_0^s(\cdot) = Q_0^\nu(\cdot \mid S_1 = s)$, and S_1 is a function of $\mathbf{V}$, the result is immediate from the presentation of Q_0^ν as the law of the pseudo-bridge (5.b) and standard Itô excursion theory. (Alternatively: note that under Q_0^s, the process $(T_w, 0 \leq w \leq s)$ is a subordinator conditioned on $T_s = 1$, which is a process with exchangeable increments. And $U_n = S_n/s$, where S_n is the local time at which the nth largest jump of this process occurs. Now use the canonical representation of processes with exchangeable increments due to Kallenberg [**19**].)

The corresponding situation under Q is only a little more complicated. Under Q there is almost surely a last interval

$$(G_1, 1) = (L_N, R_N)$$

for a random index N. This N is revealed in the $\mathbf{U}$ sequence as the unique n such that $U_n = 1$. And according to Theorem 1.2,

$$Q(N = n \mid \mathbf{V}) = V_n, \quad \text{for } n = 1, 2, \ldots .$$

Now to be given $\mathbf{V}$ and N is the same as to be given $\mathbf{V}(G_1)$ and G_1. And we know that under Q the process $(X_t, 0 \leq t \leq G_1)$ given G_1 is just a Brownian bridge of length G_1. So essentially we are being given G_1 and the interval lengths for this bridge which come from the sequence $\mathbf{V}$ by deleting the term V_N. By the previous proposition for bridges we therefore obtain the following, first for $\mu = \nu$, the Q distribution of S_1, so $Q^\nu = Q$, then for general μ by first conditioning on S_1, then mixing:

PROPOSITION 6.3. *Under Q^μ, for any μ, there is a unique random index N such that $U_N = 1$,*

$$Q^\mu(N = n \mid \mathbf{V}) = V_n, \quad \text{for } n = 1, 2, \ldots,$$

and the $\mathbf{U}$ *sequence with the* Nth *term deleted,*

$$U_1, U_2, \ldots, U_{N-1}, U_{N+1}, U_{N+2}, \ldots$$

is a sequence of independent uniform $[0, 1]$*-random variables, independent of* $\mathbf{V}$ *and* N.

Informally, Propositions 6.1, 6.2 and 6.3 tell us how the intervals forming

$$\{t\colon\ 0<t<1,\ X_t\neq 0\}$$

are created under Q^μ and Q_0^μ. According to Proposition 6.1, the ranked lengths are created with the same joint distribution in the two cases. According to Proposition 6.2, in the case of Q_0^μ these ranked lengths appear in a *purely random order*, the longest equally likely to be before or after the second longest, the 1st and 2nd and 3rd longest equally likely to be in any of the 3! possible orders, and so on. (The formula for $\mathbf{R}$ in terms of $\mathbf{V}$ and the relative local times $\mathbf{U}$, and the i.i.d. property of $\mathbf{U}$, is just a slick formalisation of this idea.)

According to Proposition 6.3, in the case of Q^μ the only difference is that one of the lengths in $\mathbf{V}$ is picked out by length-biased sampling to be the last interval $(G_1, 1)$; then the remaining intervals are laid down to the left of $(G_1, 1)$, in purely random order, relative to each other. Comparison of these two prescriptions suggests a natural way of creating the interval structure under Q^μ from that under Q_0^μ by a simple random shuffling scheme: on a suitable probability space $(\Omega, \mathbf{F}, P)$, let X have distribution Q_0^μ, and let T be a uniform $[0,1]$-random variable independent of X. Let (G_T, D_T) denote the zero-free interval of X that covers T, so

$$(G_T, D_T) = (L_N, R_N),$$

where N is the index at which $D_T - G_T$ appears in the terms of $\mathbf{V} = \mathbf{V}(X)$. Then by construction

$$P(N = n \mid \mathbf{V}) = V_n, \quad \text{for } n = 1, 2, \ldots .$$

From the interval structure $(\mathbf{V}, \mathbf{R})$ of X, create a new interval structure $(\mathbf{V}, \mathbf{R}^*)$ with the same lengths as follows: exchange the interval (G_T, D_T) with the pattern of intervals from D_T to 1. Thus

$$R_n^* = \begin{cases} R_n & \text{if } R_n < R_N \ \text{(i.e. } U_n < U_N), \\ 1 & \text{if } R_n = R_N \ \text{(i.e. } n = N,\ U_n = U_N), \\ R_n^* - V_N & \text{if } R_n > R_N \ \text{(i.e. } U_n > U_N). \end{cases}$$

These are the right ends of the zero-free intervals defined by the process

$$X^*(t) = \begin{cases} X(t) & \text{if } 0 \leq t < G_T, \\ X(D_T + t - G_T) & \text{if } G_T \leq t < 1 - (D_T - G_T), \\ X(G_T + t - [1 - (D_T - G_T)]) & \text{if } 1 - (D_T - G_T) \leq t \leq 1. \end{cases}$$

Clearly, $\mathbf{V}(X^*) = \mathbf{V}(X)$. Now it is clear that the variables $U_1, U_2, \ldots$ defined earlier on the canonical space via local times can be composed with X^* to create $U_1^*, U_2^*, \ldots$, by using, for example, the formula for local time in terms of number of intervals with lengths greater than ε. And it is clear by construction that the P-joint distribution of $\mathbf{V}(X^*)$ and $\mathbf{U}(X^*)$ is identical to the Q^μ-joint distribution of $\mathbf{V}$ and $\mathbf{U}$. Consequently, the structure of intervals forming $\{t\colon\ X_t \neq 0\}$ under Q^μ is identical to the structure of intervals forming $\{t\colon\ X_t^* \neq 0\}$ under P, where X^* is the random rearrangement of X with distribution Q_0^μ, as above. To lift this identity to the level of processes, introduce the Brownian scaling notation

$$X_{[a,b]}(u) = \frac{1}{\sqrt{(b-a)}} X_{a+u(b-a)}, \quad \text{for } 0 \leq u \leq 1,$$

for any process X and random times $a < b$. Let SSE and SSM stand for standard signed excursion and signed meander on $[0, 1]$. Let $\mathbf{Z}$ be the σ-field generated by the zero set, meaning

$$\mathbf{Z} = \sigma(\mathbf{V}, \mathbf{R}) = \sigma(\mathbf{L}, \mathbf{R}) = \sigma(\mathbf{V}, \mathbf{U})$$

(where the last of these equalities is true only Q^μ and Q_0^μ a.s. for any μ, because the definition of $\mathbf{U}$ involves local times). Standard excursion theory tells us that for any μ:

Under Q_0^μ given $\mathbf{Z}$, the processes $X_n(\cdot) = X_{[L_n, R_n]}(\cdot)$ for $n = 1, 2, \ldots$ are independent SSEs.

Under Q_μ given $\mathbf{Z}$, if N denotes the index with $R_N = 1$,

$$X_N(\cdot) = X_{[G_1, 1]}(\cdot)$$

is a SSM, and the sequence of processes $X_n(\cdot)$, for $n = 1, 2, \ldots$, with the Nth term deleted, is a sequence of independent SSEs, which is independent of X_N.

Also, because under Q_0^μ or Q^μ the whole process X can be reconstructed in a measurable manner a.s. from $(\mathbf{V}, \mathbf{R})$ and the sequence of standardised fragments $X_1(\cdot), X_2(\cdot), \ldots$, to show that a process X has one of these laws, it suffices to check that X has continuous paths and the right laws for $(\mathbf{V}, \mathbf{R})$ and the $X_n(\cdot)$, as described above. We therefore obtain the following conclusion:

THEOREM 6.4. *Let X have distribution Q_0^μ, and let T, independent of X, be uniform on $[0, 1]$. Let (G_T, D_T) be the zero-free interval for X covering T. Let M be a SSM independent of both X and T. Define a new process X^{**} by*

$$X_t^{**} = \begin{cases} X(t) & \text{if } 0 \leq t \leq G_T, \\ X(D_T + t - G_T) & \text{if } G_T \leq t \leq 1 - (D_T - G_T), \\ \sqrt{(D_T - G_T)}\, M\left(\dfrac{1}{D_T - G_T}[t + (D_T - G_T) - 1]\right) & \\ & \text{if } 1 - (D_T - G_T) \leq t \leq 1. \end{cases}$$

Then the law of X^{**} is Q^μ.

Note. On the interval $[0, G_T + 1 - D_T]$, the process X^{**} is identical to X^* defined previously. So

$$(6.d) \qquad (X_t^*, 0 \leq t \leq G_T + 1 - D_T\,;\, Q_0^\mu) \stackrel{d}{=} (X_t, 0 \leq t \leq G_1\,;\, Q^\mu).$$

In the case where $\mu = \nu$ this is Theorem 1.3.

7. *More general subordinators*

This section presents some extensions for more general subordinators of the previous results in the stable case. To go beyond the stable case, it seems essential to formulate things in a way that does not involve scaling. Previous results of this kind are Theorems 1.2, 5.1, and 5.2. The main result of this section, Theorem 7.1 below, combines all three of these results in a more general setting.

We first introduce some notation for various functions associated with a subordinator. To facilitate comparison with previous results, we use the notation '$\stackrel{*}{=}$' below to denote 'equals in the stable case'. Suppose given a subordinator (T_s) with Lévy measure Λ such that $\Lambda(0,\infty)=\infty$, and no drift component, so for $\theta>0$,

$$E[e^{-\theta T(s)}] = \exp\left(-s\int_0^\infty (1-e^{-\theta x})\Lambda(dx)\right) \stackrel{*}{=} \exp(-cs\theta^\alpha).$$

Suppose Λ has a density

$$\rho(x) = \frac{\Lambda(dx)}{dx} \stackrel{*}{=} \frac{c\alpha}{\Gamma(1-\alpha)}\frac{1}{x^{\alpha+1}}, \quad \text{for } 0<x<\infty.$$

Assume ρ is strictly positive except perhaps on an interval of the form $[a,\infty)$ for some $a>0$. The functions

$$\bar{\Lambda}(x) = \Lambda(x,\infty) \stackrel{*}{=} \frac{cx^{-\alpha}}{\Gamma(1-\alpha)},$$
$$h(x) = \bar{\Lambda}(x)/\rho(x) \stackrel{*}{=} x/\alpha$$

will play an important role. So too will the density functions

$$p(t,s) = P(S_t \in ds)/ds,$$
$$\theta(s,t) = P(T_s \in dt)/ds.$$

In principle, either of these functions can be calculated from the other, due to the inverse relation between (S_t) and (T_s). Neither function assumes a simple form in the stable case, unless $\alpha=\frac{1}{2}$. But their ratio is simple in the stable case, whatever α, due to (2.c):

$$\Phi(t,s) := \frac{\theta(s,t)}{p(t,s)} \stackrel{*}{=} \alpha\frac{s}{t}.$$

Bridge distributions $(P_{0t},\ t\geq 0)$ can be identified, as by Kallenberg in [**21**], as the family of Palm distributions on the basic probability space $(\Omega,\mathbf{F},P)$ associated with the local-time random measure dS_t:

$$P_{0t}(F) = \frac{E(dS_t\,;F)}{E(dS_t)}, \quad \text{where } F\in\mathbf{F}, t>0.$$

As suggested by [**21**, Lemma 4.1], the basic switching identity

$$P(\cdot \mid T_s = t) = P_{0t}(\cdot \mid S_t = s) \tag{7.a}$$

holds in this setting modulo null sets. We shall assume for simplicity that good versions of the conditional distributions exist so that (7.a) holds identically in s and t. Then (7.a) implies that

$$P_{0t}(S_t\in ds)/ds = \frac{\theta(s,t)}{m(t)} \stackrel{*}{=} \frac{c\Gamma(\alpha+1)s}{t^\alpha}p(t,s) \tag{7.b}$$

where

$$m(t) = \int_0^\infty \theta(s,t)\,ds = \frac{E(dS_t)}{dt} \stackrel{*}{=} \frac{t^{\alpha-1}}{c\Gamma(\alpha)}. \tag{7.c}$$

THEOREM 7.1. *Let $\mathbf{G}_t$ be the σ-field generated by the sequence of lengths*

$$V_1(t) > V_2(t) > \dots$$

of maximal intervals in the random set $Z^c \cap (0, t)$, where Z is the closure of the range of a subordinator (T_s) as above. Let $\sup\{Z \cap (0, t)\} = t - A_t$, and let $H_t = \sum_n h(V_n(t))$. Then, for $n = 1, 2, \dots$,

$$P(A_t = V_n(t) \mid \mathbf{G}_t) = h(V_n(t))/H_t. \tag{7.d}$$

The unconditional law P, and the bridge law P_{0t} are mutually absolutely continuous on $\mathbf{G}_t$, with

$$\frac{dP}{dP_{0t}} \Bigg| \mathbf{G}_t = m(t)H_t/S_t. \tag{7.e}$$

Similarly, for the laws P_t^s and P_{0t}^s obtained by further conditioning on $S_t = s$,

$$\frac{dP_t^s}{dP_{0t}^s} \Bigg| \mathbf{G}_t = \Phi(t, s)H_t/s \tag{7.f}$$

and (7.d) *holds also with P_t^s in lieu of P.*

Proof. Part (7.f) together with (7.d) for P_t^s instead of P follow immediately from Lemmas 7.2 and 7.3 below. To derive (7.e) from (7.f), compute as follows: for $G \in \mathbf{G}_t$,

$$\begin{aligned} P(G\,;\, S_t \in ds) &= P_t^s(G)p(t, s)\,ds \\ &= P_{0t}^s(GH_t)\Phi(t, s)s^{-1}p(t, s)\,ds \quad \text{(by (7.f))} \\ &= P_{0t}^s(GH_t)\theta(s, t)s^{-1}\,ds \\ &= P_{0t}(GH_t, S_t \in ds)m(t)s^{-1}\,ds \quad \text{(by (7.b)).} \end{aligned}$$

LEMMA 7.2. *Let N_t be the point process on $(0, \infty)$ which counts the interval lengths up to time $G_t = t - A_t$:*

$$N_t(\cdot) = \#\{n\colon\ V_n(t) \in \cdot\} - 1(A_t \in \cdot).$$

Let Q_t^s be the law of a Poisson point process N with intensity $s\Lambda(dx)$ over $x \in (0, \infty)$, conditioned on $\int xN(dx) = t$. Then

$$P_t^s(N_t \in dn) = Q_t^s(dn), \tag{7.g}$$

$$P_t^s(A_t \in da,\ N_t \in dn) = da\bar{\Lambda}(a)\frac{\theta(s, t-a)}{p(t, s)}Q_{t-a}^s(dn). \tag{7.h}$$

Proof. Formula (7.g) follows at once from the switching identity (7.a) and the Poisson character of the jumps of the subordinator (T_s). To derive (7.h), we recall the well-known facts that

$$P(A_t \in da) = m(t-a)\,da\bar{\Lambda}(a),$$

and that under P, given $A_t = a$, the process on $[0, t-a]$ is distributed as under the bridge law $P_{0,t-a}$. See, for example, **[21]**. This enables us to compute

$$\begin{aligned} P(A_t \in da,\ N_t \in dn,\ S_t \in ds) &= m(t-a)\,da\bar{\Lambda}(a)P_{0,t-a}(N_{t-a} \in dn,\ S_{t-a} \in ds) \\ &= m(t-a)\,da\bar{\Lambda}(a)P_{0,t-a}(S_{t-a} \in ds)Q_{t-a}^s(dn), \end{aligned}$$

where Q_u^s appears as the P_{0u}^s distribution of N_u, by (7.g). This expression and (7.b) yield (7.h).

LEMMA 7.3. *Let M be a* PPP(μ), *Σ a function such that $M\Sigma$ has probability density $\theta(t)$, with $t>0$, and let (Q_t) be a regular conditional distribution for M given $M\Sigma = t$, with $t>0$. Suppose defined on $(\Omega, \mathbf{F}, P)$ a point process N and a random variable A such that*

$$P(A \in da,\ N \in dn) = g(a)\mu(da)Q_{t-\Sigma(a)}(dn).$$

Let $N^+ = N + \delta_A$. Then $P(N^+ \in dn) = (nh_t)Q_t(dn)$, where

$$h_t(a) = g(a)\theta(t)/\theta(t - \Sigma(a)),$$

and A is picked from N^+ by h_t-biased sampling:

$$P(A \in da \mid N^+ = n) = h_t(a)n(da)/nh_t.$$

Proof. Conditioning a point process n to have sum $n\Sigma = t$, and a point at a, is the same as conditioning n to have a point at a, and $n - \delta_a$ to have sum $t - \Sigma(a)$. Let Q denote the law of a PPP(μ). From the description (2.q) of the Palm distributions for Q, it follows that for n with distribution $Q_t = Q(\cdot \mid n\Sigma = t)$, $Q_{t-\Sigma(a)}$ serves as the Palm distribution of $n - \delta_a$ given n with a point at a. The mean measure for Q_t is μ_t where

$$\begin{aligned}\mu_t(da) = Q_t n(da) &= Q[n(da) \mid n\Sigma = t] \\ &= \frac{Q[n(da)1(n\Sigma \in dt)]}{Q(n\Sigma \in dt)} \\ &= \mu(da)\theta(t - \Sigma(a))/\theta(t).\end{aligned}$$

Thus the assumptions on N and A make

$$P(A \in da,\ N \in dn) = h_t(a)\mu_t(da)Q_{t-\Sigma(a)}(dn).$$

By the above description of the Palm measures associated with Q_t and the remark below Lemma 2.2, this implies that

$$P(A \in da,\ N^+ \in dn) = h_t(a)Q_t(dn)n(da).$$

COROLLARY 7.4. *Let $\delta(t) = t\rho(t)/\bar{\Lambda}(t)$. Then for $G_t \in \mathbf{G}_t$,*

$$P_t^s[G_t\delta(A_t)] = P_{0t}^s(G_t)\Phi(t, s)t/s = P_{0t}^s(G_t)P_t^s[\delta(A_t)].$$

Proof. The first equality is immediate from (7.f) and the fact that $\sum_n V_n(t) = t$ a.s. The second equality follows by two applications of the first one.

REMARKS. In the stable case, $h(t) = t/\alpha$. Hence also $H_t = t/\alpha$, and (7.e) shows that $[dP/dP_{0t}] = m(t)t/(\alpha S_t)$ on $\mathbf{G}_t$, which is the conclusion of Theorem 5.2. Consequently, in the stable case, for $G_t \in \mathbf{G}_t$,

$$P(G_t \mid S_t) = P_{0t}(G_t \mid S_t), \tag{7.i}$$

as asserted by Theorem 5.1. The above corollary offers one kind of extension of this identity to the general case. Another extension, immediate from (7.e), is

$$P(G_t \mid S_t) = \frac{P_{0t}(G_t H_t \mid S_t)}{P_{0t}(H_t \mid S_t)}, \tag{7.j}$$

where $H_t = \sum_n h(V_n(t))$.

Converses to Theorems 1.1 *and* 1.2. We conclude with some remarks on characterisation of stable subordinators via the conclusions of Theorems 1.1 and 1.2. We would like to thank Pat Fitzsimmons for a communication in this connection, which we have incorporated in the following remarks.

Starting with Theorem 1.2, it seems reasonable to conjecture that the conclusion of this result is characteristic of the stable case. At least under the regularity conditions imposed at the beginning of this section, this follows from Theorem 7.1. Indeed, it is not hard to show that (7.d) forces $h(x) = cx$ on $\{x\colon \rho(x) > 0\}$, for some constant c; then that $\{x\colon \rho(x) > 0\} = (0, \infty)$, and finally that $h(x) = cx$ for all $x > 0$. So the stable case must obtain.

Turning to Theorem 1.1, we see that the conclusion of this theorem is the conjunction of the following three assertions:

(7.i) *the distribution of* $\mathbf{V}(t)/t$ *does not depend on* t;

(7.j) *the distribution of* $\mathbf{V}(T_s)/T_s$ *does not depend on* s;

(7.k) $\mathbf{V}(t)/t \overset{d}{=} \mathbf{V}(T_s)/T_s$ *for some* t *and* s.

Both (7.i) and (7.j) are elementary consequences of stability, and each is characteristic of the stable case, as we will indicate below. Presumably so too is the less obvious consequence of stability (7.k), but we do not see an argument. Anyway, condition (7.k) divorced from the others seems rather artificial.

That (7.i) is characteristic of the stable case can be seen using the interval-counting construction (2.d) of S_t. For (7.i) and (2.d) imply that $\bar{\Lambda}(x)$ is regularly varying at 0, and hence that there exists $p > 0$ such that

$$S_t/t^p \overset{d}{=} S_u/u^p$$

for all $t, u > 0$. Consequently, $(S_t, t \geq 0)$ is the inverse of a stable subordinator.

That (7.j) is characteristic of the stable case can be deduced as follows. The PPP description of the jumps of a subordinator implies easily that the law of $\mathbf{V}(T_s)/T_s$ determines the law of the process $(T_u/T_s, 0 \leq u \leq s)$. This in turn determines the law of T_s up to a constant scale factor by virtue of the following lemma.

LEMMA 7.5. *For a subordinator* $(T_u, 0 \leq u \leq 1)$, *the law of* T_1 *is determined up to a constant scale factor by the law of the process* $(T_u/T_1, 0 \leq u \leq 1)$.

Proof (due to David Aldous and Steven Evans). Let $W = T_1 - T_{\frac{1}{2}}$. Since $T_u/W = (T_u/T_1)(1 - T_{\frac{1}{2}}/T_1)$, from $(T_u/T_1, 0 \leq u \leq 1)$ we can construct the process $(T_u/W, 0 \leq u \leq \frac{1}{2})$. Let N be the point process on $(0, \infty)$ recording the magnitudes of jumps of this latter process. Then N is a Cox process with directing measure (i.e. random intensity) ν given by

$$\nu(A) = \tfrac{1}{2}\Lambda(WA),$$

where Λ is the Lévy measure of T. By general point process theory (see Kallenberg [20]), the law of ν is determined by that of N. But the law of ν determines that of

$$\inf\{x:\ \nu(x, \infty) \leqslant 1\} = \frac{1}{W} \inf\{x:\ \Lambda(x, \infty) \leqslant 2\}.$$

So the distribution of W (and hence that of T_1) is determined up to a constant scale factor.

REMARK 7.6. It is not always the case that the law of T_1 is determined up to a scale factor by that of T_u/T_1 for a fixed $0 < u < 1$. Indeed, the law of $T_{\frac{1}{2}}/T_1$ is beta($\frac{1}{2}, \frac{1}{2}$), both if T_1 has exponential distribution, and if T_1 is stable with index $\frac{1}{2}$. These cases are distinguished by the law of T_u/T_1 for any u other than $\frac{1}{2}$.

References

1. M. BARLOW, J. PITMAN, and M. YOR, 'On Walsh's Brownian motions', *Séminaire de Probabilités XXIII*, Lecture Notes in Mathematics 1372 (Springer, Berlin, 1989), pp. 275–293.
2. M. BARLOW, J. PITMAN, and M. YOR, 'Une extension multidimensionnelle de la loi de l'arc sinus', *Séminaire de Probabilités XXIII*, Lecture Notes in Mathematics 1372 (Springer, Berlin, 1989), pp. 294–314.
3. P. BIANE, J. F. LE GALL, and M. YOR, 'Un processus qui ressemble au pont Brownien', *Séminaire de Probabilités XXI*, Lecture Notes in Mathematics 1247 (Springer, Berlin, 1987), pp. 270–275.
4. N. H. BINGHAM and R. A. DONEY, 'On higher-dimensional analogues of the arc-sine law', *J. Appl. Probab.* 25 (1988) 120–131.
5. D. BLACKWELL and J. B. MACQUEEN, 'Ferguson distributions via Pòlya urn schemes', *Ann. Statist.* 1 (1973) 353–355.
6. J. K. BROOKS and R. V. CHACON, 'Diffusions as a limit of stretched Brownian motions', *Adv. in Math.* 49, No. 2 (1983) 109–122.
7. D. J. DALEY and D. VERE-JONES, *An introduction to the theory of point processes* (Springer, Berlin, 1988).
8. P. DONNELLY and P. JOYCE, 'Continuity and weak convergence of ranked and size-biased permutations on the infinite simplex', *Stochastic Process. Appl.* 31 (1989) 89–103.
9. E. B. DYNKIN, 'Some limit theorems for sums of independent random variables with infinite mathematical expectations', *IMS-AMS Selected Trans. Math. Statist. Prob.* 1 (1961) 171–189.
10. W. J. EWENS, 'Population genetics theory—the past and the future', *Mathematical and statistical problems in evolution* (ed. S. Lessard, University of Montreal Press, 1988), pp. 57–88.
11. T. S. FERGUSON, 'A Bayesian analysis of some nonparametric problems', *Ann. Statist.* 1 (1973) 209–230.
12. W. FELLER, *An introduction to probability theory and its applications*, Vol. 2, 2nd edn (Wiley, New York, 1971).
13. J. FRANCHI, 'Produits semi-directs de diffusions réelles et lois asymptotiques', *Adv. Appl. Probab.* 21 (1989) 756–769.
14. R. K. GETOOR, 'Excursions of a Markov process', *Ann. Probab.* 7 (1979) 244–266.
15. P. GREENWOOD and J. PITMAN, 'Fluctuation identities for Lévy processes and splitting at the maximum', *Adv. Appl. Probab.* 12 (1980) 893–902.
16. J. M. HARRISON and L. A. SHEPP, 'On skew Brownian motion', *Ann. Probab.* 9 (1981) 309–313.
17. F. M. HOPPE, 'The sampling theory of neutral alleles and an urn model in population genetics', *J. Math. Biol.* 25 (1987) 123–159.
18. K. ITÔ and H. P. MCKEAN, *Diffusion processes and their sample paths* (Springer, Berlin, 1965).
19. O. KALLENBERG, 'Canonical representations and convergence criteria for processes with exchangeable increments', *Z. Wahrsch. verw. Gebiete* 27 (1973) 23–36.
20. O. KALLENBERG, *Random measures*, 3rd edn (Akademie Verlag, Berlin, and Academic Press, London, 1983).
21. O. KALLENBERG, 'Splitting at backward times in regenerative sets', *Ann. Probab.* 9 (1981) 781–799.
22. I. KARATZAS and S. E. SHREVE, *Brownian motion and stochastic calculus* (Springer, Berlin, 1987).

23. I. KARATZAS and S. E. SHREVE, 'A decomposition of the Brownian path', *Statist. Probab. Lett.* 5 (1987) 87–93.
24. J. KEILSON and J. A. WELLNER, 'Oscillating Brownian motion', *J. Appl. Probab.* 15 (1978) 300–310.
25. J. F. C. KINGMAN, 'Random discrete distributions', *J. Roy. Statist. Soc.* 37 (1975) 1–22.
26. F. B. KNIGHT, 'On the duration of the longest excursion', *Seminar on stochastic processes, 1985* (eds E. Cinlar, K. L. Chung, and R. K. Getoor, Birkhäuser, Basel, 1985), pp. 117–148.
27. J. LAMPERTI, 'An occupation time theorem for a class of stochastic processes', *Trans. Amer. Math. Soc.* 88 (1958) 380–387.
28. J. F. LE GALL, 'One-dimensional stochastic differential equations involving the local times of the unknown process', *Stochastic analysis and applications* (eds A. Truman and D. Williams), Lecture Notes in Mathematics 1095 (Springer, Berlin, 1984), pp. 51–82.
29. P. LÉVY, 'Sur certains processus stochastiques homogènes', *Compositio Math.* 7 (1939) 283–339.
30. J. W. MCCLOSKEY, 'A model for the distribution of individuals by species in an environment', unpublished Ph.D. thesis, Michigan State University, 1965.
31. S. A. MOLCHANOV and E. OSTROVSKI, 'Symmetric stable processes as traces of degenerate diffusion processes', *Theor. Prob. Appl.* 14, No. 1 (1969) 128–131.
32. G. P. PATIL and C. TAILLIE, 'Diversity as a concept and its implications for random communities', *Proc. 41st. I.S.I. New Delhi,* 1977, pp. 497–515.
33. E. PERKINS, 'Local time is a semi-martingale', *Z. Wahrschein. verw. Gebiete* 60 (1982) 79–117.
34. M. PERMAN, 'Random discrete distributions derived from subordinators', Ph.D. thesis, Department of Statistics, University of California at Berkeley, 1990.
35. M. PERMAN, J. PITMAN, and M. YOR, 'Size-biased sampling of Poisson point processes and excursions', *Probab. Th. Rel. Fields* (1992) to appear.
36. J. PITMAN and M. YOR, 'Asymptotic laws of planar Brownian motion', *Ann. Probab.* 14 (1986) 733–779.
37. D. REVUZ and M. YOR, *Continuous Martingales and Brownian motion* (Springer, Berlin, 1991).
38. L. C. G. ROGERS, 'Williams' characterisation of the Brownian excursion law: proof and applications', *Séminaire de Probabilités XV,* Lecture Notes in Mathematics 850 (Springer, Berlin, 1981), pp. 227–250.
39. L. C. G. ROGERS, 'A guided tour through excursions', *Bull. London Math. Soc.* 21 (1989) 305–341.
40. L. C. G. ROGERS and D. WILLIAMS, *Diffusions, Markov processes and Martingales,* Vol. 2 (Wiley, New York, 1987).
41. W. ROSENKRANTZ, 'Limit theorems for solutions to a class of stochastic differential equations', *Indiana Math. J.* 24 (1975) 613–625.
42. L. A. SHEPP and S. P. LLOYD, 'Ordered cycle lengths in a random permutation', *Trans. Amer. Math Soc.* 121 (1966) 340–357.
43. A. M. VERSHIK and A. A. SCHMIDT, 'Limit measures arising in the asymptotic theory of symmetric groups I', *Theory Probab. Appl.* 22, No. 1 (1977) 70–85.
44. A. M. VERSHIK and A. A. SCHMIDT, 'Limit measures arising in the asymptotic theory of symmetric groups II', *Theory Probab. Appl.* 23, No. 1 (1978) 36–49.
45. J. B. WALSH, 'A diffusion with discontinuous local time', Astérisque 52–53 (Société Mathématique de France, Paris, 1978), pp. 37–45.
46. G. A. WATTERSON, 'The stationary distribution of the infinitely many neutral alleles diffusion model', *J. Appl. Probab.* 13 (1976) 639–651.

Department of Statistics
University of California
Berkeley
California 94720
U.S.A.

Laboratoire de Probabilités
Université Pierre et Marie Curie
4, *Place Jussieu-Tour* 56
75252 *Paris Cedex* 05
France