1. Let $X$ and $Y$ be two $\mathbb{R}^d$ valued random variables with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\text{Cov}[X] = \text{Cov}[Y]$ finite. Let $X_1, X_2, \ldots$ be independent copies of $X$ and $Y_1, Y_2, \ldots$ independent copies of $Y$. Let $U_n = n^{-1/2} \sum_{i=1}^n X_i$ and $V_n = n^{-1/2} \sum_{i=1}^n Y_i$. Is it true that for every $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ it is possible to couple $U_n$ and $V_n$ so that $\mathbb{P}[||U_n - V_n||_2 > \varepsilon] < \varepsilon$? If your answer is yes, is it true that for every $\varepsilon > 0$ there exists an $n(\varepsilon)$ such that for all $n \geq n(\varepsilon)$ it is possible to couple $U_n$ and $V_n$ so that $\mathbb{P}[U_n = V_n] \geq 1 - \varepsilon$.

**Solution:** i) The answer is **YES**.

Note that if we can couple $U_n, V_n$ such that $U_n - V_n \to 0$ in probability, then we are done. But in fact we can have almost sure convergence using Skorohod coupling (Theorem 2.2.1 in Durrett for 1-dimensional case, Theorem 3.30 in Kallenberg first edition). The theorem says that, if $X, X_1, X_2, \ldots$ is a sequence of random variable in a separable metric space such that $X_n \to X$ in distribution, then there exists a coupling $(X, X_1, X_2, \ldots)$ such that $X_n \to X$ a.s. In fact the proof goes like first constructing $X$ on a suitable space and then adding extra randomness to define $X_n$’s so that $X_n$ has the required distribution and $X_n$’s become closer to $X$ as $n \to \infty$.

In our case $U_n$ and $V_n$ both converges to multivariate normal distribution using multivariate CLT. Using the same proof as in the Skorohod coupling we have a multivariate normal r.v. $Z$ and $U_n, V_n$ such that $U_n \to Z, V_n \to Z$ a.s. And this gives the required result.

**Another proof:** One can give another proof using Skorohod embedding of random walk in Brownian motion.

ii) The second assertion is clearly not true. For example let $X$ be the random variable distributed uniformly on $\{-1, +1\}^d$ and let $Y = (Y_1, \ldots, Y_d)$ where $Y_i$’s are i.i.d standard Gaussian. Clearly $\mathbb{E}X = 0 = \mathbb{E}Y$ and $\text{Cov}[X] = \text{Cov}[Y] = I_d$. Note that for any $n$ and any coupling we have $\mathbb{P}[U_n = V_n] \leq \mathbb{P}(N(0, nI_d) \in \mathbb{Z}^d) = 0$.

2. Given a vector valued random variable $X$ taking values in $\mathbb{R}^d$ with characteristic function $\phi_X$ and an independent normal vector $Z = (Z_1, \ldots, Z_d)$ where $Z_i \sim N(0, 1)$ are independent, write the density of $X + \sigma Z$ in terms of $\phi_X$ (do not use the $d$-dim inversion formula). Deduce from it that if $\phi_X = \phi_Y$ then $X$ and $Y$ have the same distribution.

**Solution:** We need two main results to solve this problem.

i) Let $X, Y$ be two $d$-dimensional independent random variable and $a \in \mathbb{R}^d, b \in \mathbb{R}$. Let $\phi_X(\cdot), \phi_Y(\cdot)$ be corresponding characteristic functions. Then we have

$$\mathbb{E}(\phi_X(a + bY)) = \mathbb{E}(e^{ia\cdot X} \phi_Y(bX)),$$

(Use Fubini’s theorem)
ii) Let $X$ be a random vector with density $f_X$. Let $Y$ be some random variable independent of $X$. Then $X + Y$ is absolutely continuous w.r.t. Lebesgue measure and has density 

$$f_{X+Y}(t) = E(f_X(t - Y)).$$

(Write distribution of $X + Y$ in terms of that of $X$ and $Y$, change order of integration and differentiate.)

Using these two results we have 

$$f_{X+\sigma Z}(t) = E(f_{\sigma Z}(t - X)) = (2\pi\sigma^2)^{-d/2}E\phi_Z(\sigma^{-1}(t - X))$$

$$= (2\pi\sigma^2)^{-d/2}E(e^{i\sigma^{-1}t'Z}\phi_X(-\sigma^{-1}Z))$$

$$= \frac{1}{(2\pi)^d} \int e^{-it'x - \sigma^2||x||^2/2}\phi_X(x)dx.$$ 

So if we have $\phi_X = \phi_Y$, then $X + \sigma Z \overset{d}{=} Y + \sigma Z$ for every $\sigma > 0$. Let $\sigma \to 0$, using uniqueness of limit in weak convergence we have $X \overset{d}{=} Y$.

3. Let $X_n$ be a sequence of random vectors. Suppose that for every bounded $C^\infty$ function $f$ it holds that $E[f(X_n)] \to E[f(X)]$. Show that $X_n$ converges to $X$ in distribution.

Solution:

**First Proof:** (following the idea of proof of Theorem 2.2.2 in Durrett) Given $-\infty < a < b < \infty$, it is enough to find a $C^\infty$ function $f_{a,b}$ such that 

$$f_{a,b}(x) = \begin{cases} 
1 & x \leq a \\
\in [0,1] & a < x < b, \\
0 & x \geq b.
\end{cases}$$

Then use $\prod_{i=1}^d f_{a_i - \varepsilon, a_i} (x_i) \leq \prod_{i=1}^d 1\{x_i \leq a_i\} \leq \prod_{i=1}^d f_{a_i, a_i + \varepsilon} (x_i)$ to get the required result. 

W.l.o.g. we can assume $a = 0, b = 1$. Consider the function 

$$g(x) = \exp \left( -\frac{1}{x(1-x)} \right) 1_{(0<x<1)}.$$ 

Check that $g$ is a bounded $C^\infty$ function having support on $[0,1]$. Now define 

$$f_{0,1}(x) = \frac{\int x g(y)dy}{\int_0^1 g(y)dy}$$

and we are done.

**Second Proof:** Using Theorem 2.9.1 of Durrett, it is enough to prove that $E[f(X_n)] \to E[f(X)]$ for all bounded Lipschitz continuous function $f$. Given a bounded Lipschitz continuous function $f$ and $k \in \mathbb{N}$ define 

$$f_k(x) = E[f(x + k^{-1/2}Z)]$$
where $Z$ is $d$-dimensional standard Gaussian random variable. Check that $f_k$’s are bounded $C^\infty$ functions and $f_k \to f$ uniformly. Now the proof is obvious, first choose $k$ such that $f_k$ is close enough to $f$ and then use convergence for $f_k$.

4. Given a continuous function $f : [0, 1] \to \mathbb{R}$, its $n$-th Bernstein polynomial is the approximation given by:

$$f_n(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i}.$$ 

Write $f_n$ as an expected value over a probability space and use a coupling argument to show that if $f$ is increasing then so is $f_n$ for all $n$.

**Solution:** Let $U_1, U_2, \ldots, U_n$ be i.i.d. random variables having Uniform$(0,1)$ distribution. For $x \in [0,1]$ define the random variables $Z_i(x) := 1\{U_i \leq x\}$ for $i = 1,2,\ldots,n$ and $Y_x = \sum_{i=1}^{n} Z_i(x)$. Clearly $Z_i(x) \sim \text{Bin}(1,x)$ for all $x \in [0,1], i = 1,2,\ldots,n$ and $Y_x \sim \text{Bin}(n,x)$.

Now

$$\mathbb{E} f(Y_x/n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i} = f_n(x).$$

Fix $x \leq y$. Clearly we have $Z_i(x) \leq Z_i(y)$ for $i = 1,2,\ldots,n$. Hence $Y_x \leq Y_y$ a.s. Since $f$ is an increasing function we have $f_n(x) = \mathbb{E} f(Y_x/n) \leq \mathbb{E} f(Y_y/n) = f_n(y)$.

5. Give an example of a non-product measure on a product space and of two increasing functions $f$ and $g$ s.t. $\mathbb{E}[fg] < \mathbb{E}[f]\mathbb{E}[g]$.

**Solution:** Consider the probability measure $\mu$ on $\{0,1\} \times \{0,1\}$ given by $\mu\{(0,1)\} = \mu\{(1,0)\} = 1/2$. Let $f$ and $g$ be the function

- $f(0,0) = g(0,0) = 0$
- $f(1,0) = g(0,1) = 1$
- $f(0,1) = g(1,0) = 2$
- $f(1,1) = g(1,1) = 3$.

Clearly $f$ and $g$ are increasing with $\mathbb{E}[f] = \mathbb{E}[g] = 3/2$ and $\mathbb{E}[fg] = 2$. And we have $\mathbb{E}[fg] = 2 < 2.25 = \mathbb{E}[f]\mathbb{E}[g]$. 

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