Solutions to homework 9

Statistics 205B: Spring 2008

1. Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathbb{R}^d$ valued random variables with $\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{Y}] = 0$ and $\mathrm{Cov}[\mathbf{X}] = \mathrm{Cov}[\mathbf{Y}]$ finite. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent copies of $\mathbf{X}$ and $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ independent copies of $\mathbf{Y}$. Let $\mathbf{U}_n = n^{-1/2}\sum_{i=1}^n \mathbf{X}_i$ and $\mathbf{V}_n = n^{-1/2}\sum_{i=1}^n \mathbf{Y}_i$. Is it true that for every $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ it is possible to couple $\mathbf{U}_n$ and $\mathbf{V}_n$ so that $\mathbb{P}[||\mathbf{U}_n - \mathbf{V}_n||_2 > \varepsilon] < \varepsilon$? If your answer is yes, is it true that for every $\varepsilon > 0$ there exists an $n(\varepsilon)$ such that for all $n \geq n(\varepsilon)$ it is possible to couple $\mathbf{U}_n$ and $\mathbf{V}_n$ so that $\mathbb{P}[\mathbf{U}_n = \mathbf{V}_n] \geq 1-\varepsilon$.

   Solution: i) The answer is **YES**.

   Note that if we can couple $\mathbf{U}_n, \mathbf{V}_n$ such that $\mathbf{U}_n - \mathbf{V}_n \to 0$ in probability, then we are done. But in fact we can have almost sure convergence using Skorohod coupling (Theorem 2.2.1 in Durrett for 1-dimensional case, Theorem 3.30 in Kallenberg first edition). The theorem says that, if $X, X_1, X_2, \ldots$ is a sequence of random variable in a separable metric space such that $X_n \to X$ in distribution, then there exists a coupling $(X, X_1, X_2, \ldots)$ such that $X_n \to X$ a.s. In fact the proof goes like first constructing X on a suitable space and then adding extra randomness to define X_n's so that X_n has the required distribution and X_n's become closer to X as $n \to \infty$.

   In our case $\mathbf{U}_n$ and $\mathbf{V}_n$ both converges to multivariate normal distribution using multivariate CLT. Using the same proof as in the Skorohod coupling we have a multivariate normal r.v. $\mathbf{Z}$ and $\mathbf{U}_n, \mathbf{V}_n$ such that $\mathbf{U}_n \to \mathbf{Z}, \mathbf{V}_n \to \mathbf{Z}$ a.s. And this gives the required result.

   Another proof: One can give another proof using Skorohod embedding of random walk in Brownian motion.

   ii) The second assertion is clearly not true. For example let $\mathbf{X}$ be the random variable distributed uniformly on $\{-1,+1\}^d$ and let $\mathbf{Y} = (Y_1, \ldots, Y_d)$ where Y_i's are i.i.d standard Gaussian. Clearly $\mathbb{E}\mathbf{X} = \mathbf{0} = \mathbb{E}\mathbf{Y}$ and $\mathrm{Cov}[\mathbf{X}] = \mathrm{Cov}[\mathbf{Y}] = \mathbf{I}_d$. Note that for any n and any coupling we have $\mathbb{P}[\mathbf{U}_n = \mathbf{V}_n] \leq \mathbb{P}(N(\mathbf{0}, n\mathbf{I}_d) \in \mathbb{Z}^d) = 0$.

2. Given a vector valued random variable $\mathbf{X}$ taking values in $\mathbb{R}^d$ with characteristic function $\phi_{\mathbf{X}}$ and an independent normal vector $\mathbf{Z} = (Z_1, ...Z_d)$ where $Z_i \sim N(0,1)$ are independent, write the density of $\mathbf{X} + \sigma\mathbf{Z}$ in terms of $\phi_{\mathbf{X}}$ (do not use the d-dim inversion formula). Deduce from it that if $\phi_{\mathbf{X}} = \phi_{\mathbf{Y}}$ then $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution.

   Solution: We need two main results to solve this problem.

   i) Let X, Y be two d-dimensional independent random variable and $a \in \mathbb{R}^d, b \in \mathbb{R}$. Let $\phi_X(\cdot), \phi_Y(\cdot)$ be corresponding characteristic functions. Then we have

   $$\mathbb{E}(\phi_X(a + bY)) = \mathbb{E}(e^{ia'X}\phi_Y(bX)).$$

   (Use Fubini's theorem)

ii) Let X be a random vector with density f_X. Let Y be some random variable independent of X. Then $X+Y$ is absolutely continuous w.r.t. Lebesgue measure and has density

$$f_{X+Y}(t) = \mathbb{E}(f_X(t-Y)).$$

(Write distribution of $X+Y$ in terms of that of X and Y, change order of integration and differentiate.)

Using these two results we have

$$\begin{aligned} f_{\mathbf{X}+\sigma\mathbf{Z}}(\mathbf{t}) = \mathbb{E} f_{\sigma\mathbf{Z}}(\mathbf{t}-\mathbf{X}) &= (2\pi\sigma^2)^{-d/2}\mathbb{E}\phi_{\mathbf{Z}}(\sigma^{-1}(\mathbf{t}-\mathbf{X})) \\ &= (2\pi\sigma^2)^{-d/2}\mathbb{E}(e^{i\sigma^{-1}\mathbf{t}'\mathbf{Z}}\phi_{\mathbf{X}}(-\sigma^{-1}\mathbf{Z})) \\ &= \frac{1}{(2\pi)^d}\int e^{-i\mathbf{t}'\mathbf{x}-\sigma^2||\mathbf{x}||^2/2}\phi_{\mathbf{X}}(\mathbf{x})d\mathbf{x}. \end{aligned}$$

So if we have $\phi_{\mathbf{X}} = \phi_{\mathbf{Y}}$, then $\mathbf{X}+\sigma\mathbf{Z} \overset{\mathrm{d}}{=} \mathbf{Y}+\sigma\mathbf{Z}$ for every $\sigma > 0$. Let $\sigma \to 0$, using uniqueness of limit in weak convergence we have $\mathbf{X} \overset{\mathrm{d}}{=} \mathbf{Y}$.

3. Let X_n be a sequence of random vectors. Suppose that for every bounded C^∞ function f it holds that $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$. Show that X_n converges to X in distribution.

   Solution:

   First Proof: (following the idea of proof of Theorem 2.2.2 in Durrett) Given $-\infty < a < b < \infty$, it is enough to find a C^∞ function $f_{a,b}$ such that

$$f_{a,b}(x) = \begin{cases} 1 & x \le a \\ \in [0,1] & a < x < b, \\ 0 & x \ge b. \end{cases}$$

   Then use $\prod_{i=1}^d f_{a_i-\varepsilon,a_i}(x_i) \le \prod_{i=1}^d \mathbf{1}\{x_i \le a_i\} \le \prod_{i=1}^d f_{a_i,a_i+\varepsilon}(x_i)$ to get the required result. W.l.o.g. we can assume $a=0, b=1$. Consider the function

$$g(x) = \exp\left(-\frac{1}{x(1-x)}\right)\mathbf{1}_{(0<x<1)}.$$

   Check that g is a bounded C^∞ function having support on $[0,1]$. Now define

$$f_{0,1}(x) = \frac{\int_x^\infty g(y)dy}{\int_0^1 g(y)dy}$$

   and we are done.

   Second Proof: Using Theorem 2.9.1 of Durrettt, it is enough to prove that $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ for all bounded Lipschitz continuous function f. Given a bounded Lipschitz continuous function f and $k \in \mathbb{N}$ define

$$f_k(\mathbf{x}) = \mathbb{E} f(\mathbf{x} + k^{-1/2}\mathbf{Z})$$

where Z is d-dimensional standard Gaussian random variable. Check that f_k's are bounded C^∞ functions and $f_k \to f$ uniformly. Now the proof is obvious, first choose k such that f_k is close enough to f and then use convergence for f_k.

4. Given a continuous function $f : [0,1] \to \mathbb{R}$, its n-th Bernstein polynomial is the approximation given by:

$$f_n(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i}.$$

Write f_n as an expected value over a probability space and use a coupling argument to show that if f is increasing then so is f_n for all n.

Solution: Let $U_1, U_2, \ldots, U_n$ be i.i.d. random variables having Uniform$(0,1)$ distribution. For $x \in [0,1]$ define the random variables $Z_i(x) := \mathbf{1}\{U_i \le x\}$ for $i = 1, 2, \ldots, n$ and $Y_x = \sum_{i=1}^n Z_i(x)$. Clearly $Z_i(x) \sim \text{Bin}(1, x)$ for all $x \in [0,1], i = 1, 2, \ldots, n$ and $Y_x \sim \text{Bin}(n, x)$. Now

$$\mathbb{E} f(Y_x/n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i} = f_n(x).$$

Fix $x \le y$. Clearly we have $Z_i(x) \le Z_i(y)$ for $i = 1, 2, \ldots, n$. Hence $Y_x \le Y_y$ a.s. Since f is an increasing function we have $f_n(x) = \mathbb{E} f(Y_x/n) \le \mathbb{E} f(Y_y/n) = f_n(y)$.

5. Give an example of a non-product measure on a product space and of two increasing functions f and g s.t. $\mathbb{E}[fg] < \mathbb{E}[f]\mathbb{E}[g]$.

Solution: Consider the probability measure μ on $\{0,1\} \times \{0,1\}$ given by $\mu\{(0,1)\} = \mu\{(1,0)\} = 1/2$. Let f and g be the function

$$\begin{aligned} f(0,0) &= g(0,0) = 0 \\ f(1,0) &= g(0,1) = 1 \\ f(0,1) &= g(1,0) = 2 \\ f(1,1) &= g(1,1) = 3. \end{aligned}$$

Clearly f and g are increasing with $\mathbb{E}[f] = \mathbb{E}[g] = 3/2$ and $\mathbb{E}[fg] = 2$. And we have $\mathbb{E}[fg] = 2 < 2.25 = \mathbb{E}[f]\mathbb{E}[g]$.