Solutions to homework 7

Statistics 205B: Spring 2008

1. **Reading exercise:** Read (and understand) Example 6.1.5, Theorem 6.1.2 and Theorem 6.1.3 in Durrett. Give one instance where we have used Theorem 6.1.2 and one where we have used Theorem 6.1.3.

   Solution:

   (a) Theorem 6.1.2 is used in the first step of proving the Shannon-McMillam-Breiman theorem, in the construction of $X_{-1}, X_{-2}, X_{-3}, \ldots$.

   (b) Theorem 6.1.3 is used in the second step of proving the Shannon-McMillam-Breiman Theorem, when applying the Birkhoff's ergordic theorem on

$$-\frac{1}{n}\log p(X_0,\ldots,X_{n-1}|X_{-1},X_{-2},\ldots) \to \mathbb{E}(-\log p(X_0|X_{-1},X_{-2},\ldots))$$

   and

$$-\frac{1}{n}\log p^k(X_0,\ldots,X_{n-1}) \to \mathbb{E}(-\log p(X_0|X_{-1},\ldots,X_{-k})).$$

2. (Problem 7.1 from section 6.7 in Durrett)

   Given a rate one Poisson process in $[0,\infty)\times[0,\infty)$, let (X_1,Y_1) be the point that minimizes $x+y$. Let (X_2,Y_2) be the point in $[X_1,\infty)\times[Y_1,\infty)$ that minimizes $x+y$ and so on. Use this construction to show that $\gamma \geq (8/\pi)^{1/2} > 1.59$.

   Solution: It is easy to see that

$$\begin{aligned}\mathbb{P}(X_1+Y_1>t) &= \mathbb{P}(\text{there is no point inside the triangle with vertices } (0,0),(0,t),(t,0))\\ &= \mathbb{P}(Z=0),\ \text{where } Z\sim \text{Poi}(t^2/2)\\ &= e^{-t^2/2}.\end{aligned}$$

Hence we have $\mathbb{E}(X_1+Y_1) = \int_0^\infty \mathbb{P}(X_1+Y_1>t)\,dt = \int_0^\infty e^{-t^2/2}\,dt = \sqrt{\pi/2}$. Clearly $X_1 \overset{\mathrm{d}}{=} Y_1$, hence $\mathbb{E}X_1 = \sqrt{\pi/8}$. Using strong Markov property one can easily see that $(X_i - X_{i-1}, Y_i - Y_{i-1})_{i\geq 1}$ are i.i.d. Hence by Strong law of large numbers we have

$$\frac{1}{n}(X_n,Y_n) = \left(\frac{1}{n}\sum_{i=1}^n (X_i - X_{i-1}), \frac{1}{n}\sum_{i=1}^n (Y_i - Y_{i-1})\right) \to \left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) \text{ a.s.}$$

Hence $n^{-1}Z_n \to \sqrt{\pi/8}$ where $Z_n = \max\{X_n, Y_n\}$. Recall that $n^{-1}Y_{0,n} \to \gamma$ a.s. where $Y_{0,n}$ is the length of the longest increasing path lying in the square $[0,n]^2$. Now $Z_n \uparrow \infty$ a.s. Hence we have

$$\gamma = \lim_{n\to\infty}\frac{Y_{0,Z_n}}{Z_n} \geq \limsup_{n\to\infty}\frac{n}{Z_n} = \sqrt{\frac{8}{\pi}} > 1.595.$$

3. (Problem 7.2 from section 6.7 in Durrett)

   Let π_n be a random permutation of $\{1, 2, \ldots, n\}$ and let J_k^n be the number of subsets of $\{1, 2, \ldots, n\}$ of size k so that the associated $\pi_n(j)$ form an increasing subsequence. Compute $\mathbb{E}J_k^n$ and take $k \sim \alpha n^{1/2}$ to conclude that $\gamma \leq e$.

   Note: One can improve the upper bound, by noting that $\ell(\pi_n) \geq l$ implies $J_k^n \geq \binom{l}{k}$. Then use $l \sim \beta n^{1/2}, k \sim \alpha n^{1/2}$ and optimize over α. In the above case we are using $\alpha = \beta$. This was Kingman's solution. Markov inequality, Borel-Cantelli lemma and Stirling's approximation will be needed.

   Solution: Since there are $\binom{n}{k}$ subsets of $[n]$ of size k and each is in the correct order with probability $(k!)^{-1}$ we have

$$\mathbb{E}J_k^n = \frac{1}{k!}\binom{n}{k} = \frac{n!}{k!^2(n-k)!} = \frac{e^k}{\sqrt{2\pi}k}\left(\frac{k^2}{n}\right)^{-k}\left(1-\frac{k}{n}\right)^{-n+k-1/2} e^{\Theta(n^{-1})}$$

   using Stirling's approximation $n! = \sqrt{2\pi}n^{n+1/2}e^{-n+\Theta(n^{-1})}$. Now for $k \approx \alpha\sqrt{n}$ we have

$$\frac{1}{\sqrt{n}}\log \mathbb{E}J_k^n \overset{n\to\infty}{\longrightarrow} 2\alpha(1-\log\alpha) < 0$$

   if $\alpha > e$. Hence for $\alpha > e$ there exist $C, \varepsilon > 0$ such that for all $n \geq 1$ we have

$$\mathbb{E}J_k^n \leq Ce^{-\varepsilon\sqrt{n}}.$$

   Now by Markov inequality we have

$$\mathbb{P}(\ell(\pi_n) \geq \alpha\sqrt{n}) \leq \mathbb{P}(J_{\alpha\sqrt{n}}^n \geq 1) \leq \mathbb{E}J_{\alpha\sqrt{n}}^n \overset{n\to\infty}{\longrightarrow} 0$$

   and we know that $n^{-1/2}\ell(\pi_n) \to \gamma$ a.s. This implies that $\gamma \leq \alpha$ and α is any real number greater than e. Hence we have $\gamma \leq e$.

   Extra: In fact for any $0 < \beta \leq \alpha$ we have

$$\mathbb{P}(\ell(\pi_n) \geq \alpha\sqrt{n}) \leq \mathbb{P}\left(J_{\beta\sqrt{n}}^n \geq \binom{\alpha\sqrt{n}}{\beta\sqrt{n}}\right) \leq \binom{\alpha\sqrt{n}}{\beta\sqrt{n}}^{-1}\mathbb{E}J_{\beta\sqrt{n}}^n.$$

   So if we can show that there exists an α_0 such that for all $\alpha > \alpha_0$ we can find a $\beta \leq \alpha$ such that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\log \mathbb{E}J_{k_n}^n - \log\binom{l_n}{k_n}\right) < 0$$

   where $k_n = \beta\sqrt{n}, l_n = \alpha\sqrt{n}$ then we have $\gamma \leq \alpha_0$.

   [Note that it is not possible to get a better bound using some different $k = \beta n^{-p}$, $0 < p < 1/2$. In fact we have $k^{-1}\log\mathbb{P}(\ell(\pi_n) > l) \leq \log(e/\alpha\beta) + (1/2-p)\log n + (l/k-1)\log(1-k/l) - n/k\log(1-k/n) + o(1)$.]

One can find the above limit to be

$$\alpha\left[2x(1-\log\alpha) - x\log x + (1-x)\log(1-x)\right]$$

where $x = \beta/\alpha$. Now the above limit is negative for some $x \in (0,1]$ iff

$$\begin{aligned} 1-\log\alpha &\le \sup_{0<x\le 1}\frac{1}{2}\left[\log x - \frac{1-x}{x}\log(1-x)\right] = a(\text{say}) \\ \text{i.e. } \alpha &\ge e^{1-a}. \end{aligned}$$

Hence we have $\alpha_0 = \exp(1-a) = 1/\sqrt{x^*(1-x^*)}$ where x^* is the unique solution of $1-x = e^{-2x}$ in $(0,1]$. One can easily find approximate value for $\alpha_0 \approx 2.49$.

4. (Problem 7.4 from section 6.7 in Durrett)

   Oriented first passage percolation. Consider a graph with vertices $\{(m,n) \in \mathbb{Z}^2 : m+n \text{ is even and } n \le 0\}$, and oriented edges connecting (m,n) to $(m-1,n-1)$ and (m,n) to $(m+1,n-1)$. Assign i.i.d. exponential mean one random variables to each edge. Thinking of the number of edge e as giving the time it takes water to travel down the edge, define $t(m,n)$ = the time at which the fluid first reaches (m,n) starting from the origin $(0,0)$, and $a_n = \inf\{t(m,-n)\}$, the time needed to reach depth n below the surface. Show that as $n \to \infty$, a_n/n converges to a limit γ a.s.

   Solution: Let k_m be the integer so that $t(k_m, m) = a_m$ with $k_0 = 0$. Let $X_{m,n}$ be the amount of time it takes water starting from (k_m, m) to reach depth n. It is clear that $X_{0,m} + X_{m,n} \ge X_{0,n}$ and $a_n = X_{0,n}$. Using i.i.d.-ness of edge weights one can easily verify condition (ii) and (iii). Since $\mathbb{E}X_{0,1}^+ < \infty$ and $X_{m,n} \ge 0$ (iv) holds. (6.1) implies that $X_{0,n}/n \to X$ a.s. To see that the limit is constant, enumerate the edges in some order (e.g., take each row in turn from left to right) $e_1, e_2, \ldots$ and observe that X is measurable with respect to the tail-field of the i.i.d. sequence of edge weights $\{\tau(e_1), \tau(e_2), \ldots\}$.

5. (Problem 7.5 from section 6.7 in Durrett)

   Continue the setup in the last exercise:

   (a) Show $\gamma \le 1/2$ by considering a_1.

   (b) Get a positive lower bound on γ by looking at the expected number of paths down to $\{(m,-n) : -n \le m \le n\}$ with passage time $\le an$ and using results from Section 1.9.

   Solution:

   (a) $a_1 = X_{0,1}$ is the minimum of two mean one i.i.d. exponentials so it is a mean 1/2 exponential. Hence $\gamma = \inf n^{-1}\mathbb{E}(a_n) \le \mathbb{E}(a_1) = 1/2$.

   (b) Let S_n be the sum of n independent mean 1 exponentials. Large deviation results in Section 1.9 imply that for $a < 1$ we have

   $$\frac{1}{n}\log\mathbb{P}(S_n \le an) \longrightarrow -a + 1 + \log a.$$

Since there are 2^n paths down to level n, we see that if $f(a) = \log 2 - a + 1 + \log a < 0$ then $\limsup_{n\to\infty} n^{-1} \log \mathbb{P}(a_n \le an) \le \log 2 + \limsup_{n\to\infty} n^{-1}\mathbb{P}(S_n \le an) = f(a) < 0$. Hence using Markov inequality and Borel Cantelli lemma we have $\gamma \ge a$. Numerically we have $\gamma \ge 0.232$.