Solution to homework 10

Statistics 205B: Spring 2008

1. (Problem 1.3 from section 7.1 in Durrett)

Fix $t$ and let $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$. Compute

$$E \left( \sum_{m \leq 2^n} \Delta_{m,n}^2 - t \right)^2$$

and use Borel-Cantelli Lemma to conclude that $\sum_{m \leq 2^n} \Delta_{m,n}^2 \rightarrow t$ a.s. as $n \rightarrow \infty$.

**Solution:** For fixed $n$, clearly $\Delta_{m,n}$'s are i.i.d. $N(0, t2^{-n})$ for $1 \leq m \leq 2^n$. Hence we have $E \Delta_{m,n}^2 = t2^{-n}$ and $\text{Var}(\Delta_{m,n}^2) = 3t2^{-2n}$. Therefore we have

$$E \left( \sum_{m \leq 2^n} \Delta_{m,n}^2 - t \right)^2 = \text{Var}(\sum_{m \leq 2^n} \Delta_{m,n}^2) = \sum_{m \leq 2^n} \text{Var}(\Delta_{m,n}^2) = 3t2^{-n}.$$

Hence,

$$\sum_{n=1}^{\infty} P \left( \left| \sum_{m \leq 2^n} \Delta_{m,n}^2 - t \right| > \frac{1}{2n^{\frac{1}{4}}} \right) \leq \sum_{n=1}^{\infty} 2^{n/2} E \left( \sum_{m \leq 2^n} \Delta_{m,n}^2 - t \right)^2 = \sum_{n=1}^{\infty} 3t2^{-2n/2} < \infty.$$

By Borel-Cantelli lemma we have $P(\limsup_{n \rightarrow \infty} \left\{ \left| \sum_{m \leq 2^n} \Delta_{m,n}^2 - t \right| > \frac{1}{2n^{\frac{1}{4}}} \right\}) = 0$, which implies that $\sum_{m \leq 2^n} \Delta_{m,n}^2 \rightarrow t$ a.s. as $n \rightarrow \infty$.

2. (Problem 1 from chapter 13 in Kallenberg)

Let $\xi_1, \xi_2, \ldots, \xi_n$ be i.i.d. $N(m, \sigma^2)$. Show that the random variables $\tilde{\xi} = n^{-1} \sum_k \xi_k$ and $s^2 = (n-1)^{-1} \sum_k (\xi_k - \tilde{\xi})^2$ are independent and that $(n-1)s^2 \overset{d}{=} \sum_{k<n} (\xi_k - m)^2$.

**Hint:** Use the fact that, for i.i.d. random variable $\xi_1, \xi_2, \ldots, \xi_d$, $d \geq 2$, $(\xi_1, \xi_2, \ldots, \xi_d)$ is spherically symmetric iff $\xi_i$ are centered Gaussian.

**Solution:** Let $A$ be an orthogonal matrix with first row equal to $n^{-1/2}(1, 1, \ldots, 1)$, $A$ exists because $n^{-1/2}(1, 1, \ldots, 1)$ is an orthonormal vector and using Gram-Schmidt algorithm. For a concrete choice use the orthogonal matrix

$$A_{1,j} = n^{-1/2}, \ A_{i,j} = (i(i-1))^{-1/2}(1\{j < i\} - (i-1)1\{j = i\})$$

for $i = 2, 3, \ldots, n$. Now the co-ordinates of $\xi = (\xi_1 - m, \xi_2 - m, \ldots, \xi_n - m)'$ are i.i.d. $N(0, \sigma^2)$.

Hence by spherical symmetry we have $\eta_1, \eta_2, \ldots, \eta_n$ are i.i.d $N(0, \sigma^2)$ where $\eta_i = A_i \xi$ and $A_i$ is the $i$-th row of $A$. Note that $\eta_1 = n^{1/2}(\xi - m)$ and $\sum_{i=1}^n \eta_i^2 = \sum_{i=1}^n (\xi_i - m)^2$. Hence

$$(n-1)s^2 = \sum_k (\xi_k - \tilde{\xi})^2 = \sum_k (\xi_k - m)^2 - n(\tilde{\xi} - m)^2 = \sum_{i=1}^n \eta_i^2 - \eta_1^2 = \sum_{i=2}^n \eta_i^2.$$

From this it follows that $\xi$ and $s^2$ are independent and $(n-1)s^2 \overset{d}{=} \sum_{k<n} (\xi_k - m)^2$.
3. (Problem 3 from chapter 13 in Kallenberg)

Let $B$ be a Brownian motion on $[0,1]$, and define $X_t := B_t - tB_1$. Show that $X \perp B_1$. Use this fact to express the conditional distribution of $B$, given $B_1$, in terms of a Brownian bridge.

**Solution:** Fix $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$. Clearly we have $(B_{t_1} - t_1 B_1, B_{t_2} - t_2 B_1, \ldots, B_{t_n} - t_n B_1, B_1)$ follows a multivariate normal distribution and $\text{Cov}(B_{t_i} - t_i B_1, B_1) = 0$ for all $1 \leq i \leq n$. Hence $B_1$ is independent of $(B_{t_1} - t_1 B_1, B_{t_2} - t_2 B_1, \ldots, B_{t_n} - t_n B_1)$ for every $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$. Now $\pi - \lambda$ theorem implies that $B_1$ is independent of $(B_t - tB_1)_{0 \leq t \leq 1}$. Therefore conditional on $B_1$ the distribution of $B$ is $B_1 I + X$ where $I$ is the identity function $I(t) = t$ and $X$ is an independent Brownian Bridge.

4. Let $D$ be a dense subset of $[0,1]$ and suppose $f$ defined on $D$ is Holder-continuous with parameter $c > 0$. Show that there is a unique continuous extension of $f$ to the interval $[0,1]$ and that the extension is $c$-Holder continuous.

**Hint:** Recall that a function $f$ is $\alpha$-Holder continuous if $|f(x) - f(y)| \leq C|x - y|^{\alpha}$ for all $x, y$ and $D$ is dense in $[0,1]$ if for any $x \in [0,1]$ there is a sequence of numbers $x_n \in D$ such that $|x_n - x| \to 0$.

**Solution:** Fix $x \in [0,1]$. Since $D$ is dense in $[0,1]$, we can find a sequence $x_n \in D$ such that $|x_n - x| \to 0$. Clearly $|f(x) - f(x_n)| \leq C|x - x_n|^{c}$. Hence $f(x_n)$ is a Cauchy sequence and $\lim_{n \to \infty} f(x_n)$ exists. Define $\tilde{f}(x) = \lim_{n \to \infty} f(x_n)$. Note that $\tilde{f}(x)$ is well-defined, since the limit depends only on $x$ and not on the particular sequence $x_n$. This is clear from the fact that if $x_n \to x, y_n \to x$ then $|x_n - y_n| \to 0$ and hence $|f(x_n) - f(y_n)| \leq C|x_n - y_n|^{c} \to 0$. So $\tilde{f}$ is well-defined on $[0,1]$ and is an extension of $f$.

Now fix any two real number $x, y \in [0,1]$. Let $x_n, y_n \in D$ such that $x_n \to x$ and $y_n \to y$. Then we have $|f(x_n) - f(y_n)| \leq C|x_n - y_n|^{c}$. Taking limit as $n \to \infty$ we have $|\tilde{f}(x) - \tilde{f}(y)| \leq C|x - y|^{c}$. Hence $\tilde{f}$ is continuous, in fact $c$-Holder continuous.

Also from the definition of $\tilde{f}$ it is clear that $\tilde{f}$ is the unique continuous extension of $f$.

5. Construct a version of Brownian motion that has the same finite dimensional distributions but is a.s. discontinuous.

**Extra Points:** Do the same so that the version is a.s. nowhere continuous (i.e. everywhere discontinuous).

**Solution:** Let $B$ is a standard brownian motion on $[0,1]$ which is continuous a.s.. Let $U$ be an independent Uniform$[0,1]$ random variable. Define

$$\tilde{B}_t = B_t + 1_{\{t = U\}}.$$ 

Clearly $\tilde{B}$ has the same finite dimensional distributions as $B$. But

$$\mathbb{P}(\tilde{B} \text{ is continuous}) = \mathbb{P}(1_{\{t = U\}} \text{ is continuous}) = 0.$$
**Extra Points:** Let $D_1, D_2$ be two disjoint dense subsets of $[0, 1]$. Let $f$ be a continuous function on $[0, 1]$. Define

$$
\tilde{f}(x) = \begin{cases} 
f(x) & \text{if } x \notin D_1 \cup D_2 \\
f(x) - 1 & \text{if } x \in D_1 \\
f(x) + 1 & \text{if } x \in D_2.
\end{cases}
$$

Then $\tilde{f}$ is nowhere continuous.

Now let $D_1, D_2$ be two random subsets of $[0, 1]$ such that both are dense and disjoint a.s. and for any real number $t \in [0, 1]$, $\mathbb{P}(t \in D_1 \cup D_2) = 0$.

Define $\tilde{B}$ with $D_1, D_2$ as before. $\tilde{B}$ is nowhere continuous a.s. Also for any finite collection $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ we have $(\tilde{B}_{t_1}, \tilde{B}_{t_2}, \ldots, \tilde{B}_{t_n}) \overset{d}{=} (B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ since $\mathbb{P}((t_1, t_2, \ldots, t_n) \in D_1 \cup D_2) = 0$.

There are several ways of getting such $D_1, D_2$. One of them is to take two fixed countable disjoint dense subsets, e.g. $C_i = \{k(i + 1)^{-n} : k \in \mathbb{Z}, n \in \mathbb{N}\}, i = 1, 2$. Define $D_i = U + C_i, i = 1, 2$ where $U$ is a uniform $[0, 1]$ random variable. Or take one countable dense subset $D$ and two uniform random variable $U_1, U_2$ and let $D_i = U_i + D, i = 1, 2$. 