Homework 8
Statistics 205B: Spring 2008
Due on March 20, 2008

1. (Problem 1.12 from section 3.1 in Durrett)
   Let \( X_1, X_2, X_3, \ldots \) be i.i.d. uniform on \((0, 1)\), let \( S_n = X_1 + X_2 + \cdots + X_n \), and \( T = \inf\{n : S_n > 1\} \). Show that \( \mathbb{P}(T > n) = 1/n! \), so \( \mathbb{E}T = e \) and \( \mathbb{E}S_T = e/2 \).

2. (Problem 1.13 from section 3.1 in Durrett)
   Let \( X_1, X_2, X_3, \ldots \) be i.i.d. with \( \mathbb{P}(X_1 = 1) = p > 1/2 \) and \( \mathbb{P}(X_1 = -1) = 1 - p \), and let \( S_n = X_1 + X_2 + \cdots + X_n \). Let \( \alpha = \inf\{m : S_m > 0\} \) and \( \beta = \inf\{n : S_n < 0\} \).
   (a) Use Exercise 3.1.9 to conclude that \( \mathbb{P}(\alpha < \infty) = 1 \) and \( \mathbb{P}(\beta < \infty) < 1 \).
   (b) If \( Y = \inf S_n \), then \( \mathbb{P}(Y \leq -k) = \mathbb{P}(\beta < \infty)^k \).
   (c) Apply Wald’s equation to \( \alpha \wedge n \) and let \( n \to \infty \) to get \( \mathbb{E} \alpha = 1/\mathbb{E}X_1 = 1/(2p - 1) \).
      Comparing with exercise 1.10 shows \( \mathbb{P}(\bar{\beta} = \infty) = 2p - 1 \).

3. (Problem 1.15 from section 3.1 in Durrett)
   (Wald’s second equation.) Let \( X_1, X_2, \ldots \) be i.i.d. with \( \mathbb{E}X_n = 0 \) and \( \mathbb{E}X_n^2 = \sigma^2 < \infty \). If \( T \) is a stopping time with \( \mathbb{E}T < \infty \) then \( \mathbb{E}S_T^2 = \sigma^2 \mathbb{E}T \).
   **Hint:** Compute \( \mathbb{E}S_{T\wedge n}^2 \) by induction and show that \( S_{T\wedge n} \) is a Cauchy sequence in \( L^2 \).

4. Show that if the random walk \( S_n \) is recurrent then so is the random walk \( S_{k \times n} \) for each natural \( k \).

5. Consider a random walk \( S_n \) in \( \mathbb{R}^2 \) where \( S_n = \sum_{i=1}^n X_i \) and \( X_i \) are i.i.d. with \( \mathbb{E}[X_i(1)] = \mathbb{E}[X_i(2)] = 0 \) and \( \mathbb{E}[X_i(1)^2 + X_i(2)^2] < \infty \). Let \( U((-1,1) \times (-1,1)) \) be the occupation measures of the \((-1,1) \times (-1,1)\) square; that is, the expected value of the number of visits of the walk to the square. Show that \( U((-1,1) \times (-1,1)) = \infty \).
   From what we’ve seen in class this implies that the random walk \( S_n \) is recurrent.