Homework 8

Statistics 205B: Spring 2008
Due on March 20, 2008

1. (Problem 1.12 from section 3.1 in Durrett)

   Let $X_1, X_2, X_3, \ldots$ be i.i.d. uniform on $(0,1)$, let $S_n = X_1 + X_2 + \cdots + X_n$, and $T = \inf\{n : S_n > 1\}$. Show that $\mathbb{P}(T > n) = 1/n!$, so $\mathrm{E}\,T = e$ and $\mathbb{E}S_T = e/2$.

2. (Problem 1.13 from section 3.1 in Durrett)

   Let $X_1, X_2, X_3, \ldots$ be i.i.d. with $\mathbb{P}(X_1 = 1) = p > 1/2$ and $\mathbb{P}(X_1 = -1) = 1 - p$, and let $S_n = X_1 + X_2 + \cdots + X_n$. Let $\alpha = \inf\{m : S_m > 0\}$ and $\beta = \inf\{n : S_n < 0\}$.

   (a) Use Exercise 3.1.9 to conclude that $\mathbb{P}(\alpha < \infty) = 1$ and $\mathbb{P}(\beta < \infty) < 1$.

   (b) If $Y = \inf S_n$, then $\mathbb{P}(Y \le -k) = \mathbb{P}(\beta < \infty)^k$.

   (c) Apply Wald's equation to $\alpha \wedge n$ and let $n \to \infty$ to get $\mathbb{E}\alpha = 1/\mathbb{E}X_1 = 1/(2p-1)$. Comparing with exercise 1.10 shows $\mathbb{P}(\bar{\beta} = \infty) = 2p - 1$.

3. (Problem 1.15 from section 3.1 in Durrett)

   (Wald's second equation.) Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 = \sigma^2 < \infty$. If T is a stopping time with $\mathbb{E}T < \infty$ then $\mathbb{E}S_T^2 = \sigma^2 \mathbb{E}T$.

   Hint: Compute $\mathbb{E}S_{T\wedge n}^2$ by induction and show that $S_{T\wedge n}$ is a Cauchy sequence in L^2.

4. Show that if the random walk S_n is recurrent then so is the random walk $S_{k\times n}$ for each natural k.

5. Consider a random walk S_n in $\mathbb{R}^2$ where $S_n = \sum_{i=1}^n X_i$ and X_i are i.i.d. with $\mathbb{E}[X_i(1)] = \mathbb{E}[X_i(2)] = 0$ and $\mathbb{E}[X_i(1)^2 + X_i(2)^2] < \infty$. Let $U((-1,1)\times(-1,1))$ be the occupation measures of the $(-1,1)\times(-1,1)$ square; that is, the expected value of the number of visits of the walk to the square. Show that $U((-1,1)\times(-1,1)) = \infty$.

   From what we've seen in class this implies that the random walk S_n is recurrent.