1. Let $Z_n$ be a branching process where the number of offsprings has the same distribution as $X$ where $\text{E}[X] = \mu = 1$ and $P(X = 1) < 1$. Prove that a.s. $Z_n = 0$ for all sufficiently large $n$.

2. Let $X = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix}$ be a $2 \times 2$ matrix of non-negative integer valued random variables. The multi-type branching process defined by $X$ is defined as follows: Let $(Z_{1,1}^1, Z_{2,1}^2) = (1, 1)$ and by induction let:

$$
(Z_{n+1}^1, Z_{n+1}^2) = \left( \sum_{i=1}^{Z_{n,1}^1} X_{i,1}^{n+1,i} + \sum_{i=1}^{Z_{n,2}^1} X_{i,2}^{n+1,i}, \sum_{i=1}^{Z_{n,1}^1} X_{1,2}^{n+1,i} + \sum_{i=1}^{Z_{n,2}^1} X_{2,2}^{n+1,i} \right)
$$

where all the random variables $X_{i,j}^{k,l}$ are independent and $X_{i,j}^{k,l}$ has the same distribution as $X_{i,j}$. Let $\mu = E(X) = \begin{pmatrix} \mu_{1,1} \\ \mu_{2,1} \\ \mu_{1,2} \\ \mu_{2,2} \end{pmatrix}$ and assume that all entries of $\mu$ are positive. Let $(\nu_1, \nu_2)$ be the Perron eigenvector of $\mu$, so that

$$
\mu \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \lambda \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}
$$

where $\lambda, \nu_1, \nu_2$ are positive.

(a) Show that $\lambda^{-n}(\nu_1 Z_{n}^1 + \nu_2 Z_{n}^2)$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_{i,j}^{k,l} : k \leq n)$.

(b) Conclude that when $\lambda < 1$ then a.s. for all large $n$ it holds that $Z_{n}^1 = Z_{n}^2 = 0$.

(c) Prove the same when $\lambda = 1$.

3. Let $Z_{n+1} = \sum_{i=1}^{Z_n} X_i^{n+1}$ be a branching process, with $Z_1 = 1$. Show that $Z_n$ is a Markov chain with respect to the filtration $\mathcal{F}_n = \sigma(X_i^m : m \leq n, i \geq 1)$, $n \geq 1$.

4. (Problem 1.3 from section 5.2 in Durrett)

Suppose $S = \{0, 1\}$ and

$$
p = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}
$$

Use induction to show that

$$
P_\mu(X_n = 0) = \frac{\beta}{\alpha + \beta} + (1 - \alpha - \beta)^n \left\{ \mu(0) - \frac{\beta}{\alpha + \beta} \right\}.
$$

Homework 2
Statistics 205B: Spring 2008
Due on February 5, 2008
5. (Problem 1.8 from Section 5.2 in Durrett)

Let $\theta, U_1, U_2, \ldots$ be independent and uniform on $(0, 1)$. Let $X_i = 1$ if $U_i < \theta$, $=-1$ if $U_i > \theta$, and let $S_n = X_1 + \ldots + X_n$. In words we first pick $\theta$ according to the uniform distribution and then flip a coin with probability $\theta$ of heads to generate a random walk.

(a) Compute $P(X_{n+1} = 1|X_1, \ldots, X_n)$ and
(b) conclude $S_n$ is a temporally inhomogeneous Markov chain.