Homework 13

Statistics 205B: Spring 2008
Due on May 1, 2008

1. (Problem 4.1 from section 7.4 in Durrett)

   (a) Generalize the proof of 7.4.6 to conclude that if $u < v \le a$ then
   $$\mathbb{P}_0(T_a < t, u < B_t < v) = \mathbb{P}_0(2a - v < B_t < 2a - u).$$

   (b) Let $M_t = \max_{0 \le s \le t} B_s$. Use (a) to derive the joint density
   $$\mathbb{P}_0(M_t = a, B_t = x) = \frac{2(2a - x)}{\sqrt{2\pi t^3}} e^{-(2a-x)^2/2t}.$$

2. (Problem 6.2 from section 7.6 in Durrett)

   Suppose S_n is one-dimensional simple random walk and let
   $$R_n = 1 + \max_{m \le n} S_m - \min_{m \le n} S_m$$
   be the number of points visited by time n. Show that $R_n/\sqrt{n} \implies$ a limit.

3. (Problem 6.3 from section 7.6 in Durrett)

   If $X_1, X_2, \dots$ are i.i.d. with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$, then from example 7.6.5 we have
   $$n^{-3/2} \sum_{m=1}^{n} (n + 1 - m) X_m \implies \int_0^1 B_t dt.$$

   (a) Show that the right hand side has a normal distribution with mean 0 and variance 1/3.

   (b) Deduce the result from the Lindeberg-Feller theorem.

4. (Problem 9.2 from section 7.9 in Durrett)

   Show that if $\mathbb{E}|X_i|^\alpha = \infty$ for some $\alpha < 2$ then
   $$\limsup_{n \to \infty} |S_n|/n^{1/\alpha} = \infty \text{ a.s.}$$
   so the law of iterated logarithm fails.

   Hint: First show that $\limsup_{n\to\infty} |X_n|/n^{1/\alpha} = \infty$ a.s.

5. (Problem 9.3 from section 7.9 in Durrett)

   Give a direct proof that the limit set of $\{S_n/(2n \log\log n)^{1/2}\}$ is $[-1, 1]$, where $X_1, X_2, \dots$ are i.i.d. with $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1$ and $S_n = X_1 + \cdots + X_n$.

   Hint: Use the law of iterated logarithm to get the extreme points and then fill the middle points.