Homework 10

Statistics 205B: Spring 2008
Due on April 10, 2008

1. (Problem 1.3 from section 7.1 in Durrett)

   Fix t and let $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$. Compute

   $$\mathbb{E}\left(\sum_{m \le 2^n} \Delta_{m,n}^2 - t\right)^2$$

   and use Borel-Cantelli Lemma to conclude that $\sum_{m \le 2^n} \Delta_{m,n}^2 \to t$ a.s. as $n \to \infty$.

2. (Problem 1 from chapter 13 in Kallenberg)

   Let $\xi_1, \xi_2, \ldots, \xi_n$ be i.i.d. $\mathrm{N}(m, \sigma^2)$. Show that the random variables $\bar{\xi} = n^{-1} \sum_k \xi_k$ and $s^2 = (n-1)^{-1} \sum_k (\xi_k - \bar{\xi})^2$ are independent and that $(n-1)s^2 \stackrel{\mathrm{d}}{=} \sum_{k<n} (\xi_k - m)^2$.

   Hint: Use the fact that, for i.i.d. random variable $\xi_1, \xi_2, \ldots, \xi_d$, $d \ge 2$, $(\xi_1, \xi_2, \ldots, \xi_d)$ is spherically symmetric iff ξ_i are centered Gaussian.

3. (Problem 3 from chapter 13 in Kallenberg)

   Let B be a Brownian motion on $[0,1]$, and define $X_t := B_t - tB_1$. Show that $X \perp B_1$. Use this fact to express the conditional distribution of B, given B_1, in terms of a Brownian bridge.

   Hint: First show that $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ is independent of B_1 for any $0 \le t_1 < t_2 < \cdots < t_k \le 1$ and then extend it to the whole process X.

4. Let D be a dense subset of $[0,1]$ and suppose f defined on D is Holder-continuous with parameter $c > 0$. Show that there is a unique continuous extension of f to the interval $[0,1]$ and that the extension is c-Holder continuous.

   Hint: Recall that a function f is α-Holder continuous if $|f(x) - f(y)| \le C|x-y|^\alpha$ for all x, y and D is dense in $[0,1]$ if for any $x \in [0,1]$ there is a sequence of numbers $x_n \in D$ such that $|x_n - x| \to 0$.

   Hint: The problem says that, there is one continuous extension, it is unique and it is c-Holder continuous. So there are three parts in the problem. First construct one extension with the required properties. show that it is unique and c-Holder continuous.

5. Construct a version of Brownian motion that has the same finite dimensional distributions but is a.s. discontinuous.

   Extra Points: Do the same so that the version is a.s. nowhere continuous (i.e. everywhere discontinuous).

Hint: Example of a nowhere continuous function is $f(x) = \mathbf{1}\{x \text{ is rational}\}$. In fact there is nothing special about the set of rationals, any set $\mathcal{D} \subset [0,1]$ such that D and D^c both are dense will do. Also from the example one can easily see how to change a given continuous function at countable number of points to make it nowhere continuous. Now use the first part with these arguments to do the construction.