Lecture 4

Lecture date: Sep 7 *Scribe: Allan Sly*

1 SATs

Definition 1 *A 'SAT' formula has n variables and m constraints or clauses. The variables are denoted $\{x_i\}_{i=1}^n, x_i \in \{0,1\}$ and we denote $\overline{x}_i = 1 - x_i$ and z_i is a literal if $z_i = x_i$ or $z_i = \overline{x}_i$.*

A clause C_a is given by $C_a = z_{i_1} \vee \ldots \vee z_{i_k}$ and $C_a(x_{i_1}, \ldots, x_{i_k}) = 0$ iff $z_{i_1} = z_{i_2} \ldots = z_{i_k} = 0$ otherwise $C_a(x_{i_1}, \ldots, x_{i_k}) = 1$. Also denote $\partial a = \{i_1, \ldots, i_k\}$.

A SAT formula is given by $f(x) = C_1(x) \wedge \ldots \wedge C_m(x)$ by which we mean $f(x) = 1$ iff $C_i(x) = 1$ for all i otherwise $f(x) = 0$. This is equivalent to $f(x) = \prod_{i=1}^m C_i(x)$

If we look at $f(x) = \frac{1}{Z} \prod_{i=1}^m C_i(x)$ we get a factorization of the uniform distribution on SAT assignments x, according to the factor graph $([n], [m], \{\partial a : 1 \le a \le m\})$.

The basic questions of interest for a SAT are:

1. Is there a solution to the equation $f(x) = 1$?
2. What's the distribution of SAT assignments, can you sample from them?
3. What can be said for random formulations?

We say k-SAT means all clauses are on k-variables.

1.1 1-SAT

Claim 2 *A 1-SAT formula has a solution iff there does not exist x_i such that x_i and $\overline{x}_i$ are clauses of the formula.*

Question 3 *How do you sample from a SAT assignment?*

Sampling the SAT assignments uniformly can be done one variable at a time. When some SAT assignment exists there is either one or two possible choices for each variable. If there are two choices choose with probability 1/2.

Exercise 4 *(1 point) Consider the random 1-SAT formula ψ with n variables and $\eta\sqrt{n}$ random 1-clauses. Show that $P[\psi$ is satisfiable$] \to h(\eta)$ as $n \to \infty$ where for all η $h(\eta) \in (0,1)$ and $\lim_{\eta\to 0} h(\eta) = 1$ and $\lim_{\eta\to\infty} h(\eta) = 0$.*

Exercise 5 *(2 points) Find $h(\eta)$.*

1.2 2-SAT

We can associate a 2-SAT formula ψ with a directed graph $DG(\psi)$ with vertices $x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n$ where $z_i \to z_j$ is an edge of $DG(\psi)$ iff $\overline{z}_i \vee z_j$ is a clause of ψ. This also implies that $\overline{z}_j \to \overline{z}_i$ is an edge.

Definition 6 *The transitive closure of $DG(\psi)$ is a directed graph where $z_i \to z_j$ iff there exists a path $z_i \to z_{i_1} \to \ldots \to z_{i_k} \to z_j$.*

Claim 7

1. *If $z_i \to z_j$ is an edge of the transitive closure then for all SAT assignments with $z_i = 1$ it holds that $z_j = 1$.*
2. *ψ is SAT iff there does not exist an i such that $x_i \to \overline{x}_i$ and $\overline{x}_i \to x_i$.*

Exercise 8 *(1 or 2 points inversely proportional to proof length) Prove part 2 of the claim.*

Algorithm for 2-SAT

1. Construct $DG(\psi)$.
2. Pick x_i. If $x_i \to \overline{x}_i$ and $\overline{x}_i \to x_i$ return UNSAT.
3. If $x_i \to \overline{x}_i$ and $\overline{x}_i \not\to x_i$ then (I) set $x_i = 0$ and all $z_j = 1$ if $\overline{x}_i \to z_j$.
4. If $x_i \not\to \overline{x}_i$ and $\overline{x}_i \to x_i$ then (II) set $x_i = 1$ and all $z_j = 1$ if $x_i \to z_j$.
5. Otherwise choose between (I) and (II).
6. Repeat until all x_i are determined.

Exercise 9 *(1 point) Prove that the algorithm works.*

2nd Algorithm for 2-SAT

This algorithm assumes that there exists a SAT assignment. Pick $X = X^0$ uniformly at random. While $\psi(X^I) \neq 1$

1. Take one of the clauses $z_i \vee z_j$ UNSAT by X^I.
2. To get X^{I+1} with probability $\frac{1}{2}$ flip x_i^I otherwise flip x_j^I.

Claim Suppose that $\psi(Y) = 1$ and that $\psi(X^I) = 0$ then

$$P[d_H(Y, X^{I+1}) = d_H(Y, X^I) - 1] \geq \frac{1}{2}$$

where d_H is the hamming distance $d_H(X, Y) = \#\{i : x_i \neq y_i\}$ since at least one of x_i^I and x_j^I must be flipped in Y to satisfy $z_i \vee z_j$.

Claim 10 *Consider the random walk Z_t started at $d(X^0, Y)$ on $[0, n]$ (where at n you move to $n-1$ with probability 1). Let T be the time the algorithm terminates and let T' be the time the random walk hits 0. Then*

$$P[T \geq t] \leq P[T' \geq t].$$

Let Z_t be the random walk and $Z_0 = j = d(X^0, Y)$. As $d(X^t, Y)$ is always at least as likely to decrease as Z_t (except when $d(X^t, Y) < Z_t = n$) the two processes can be coupled so that $d(X^t, Y) \leq Z_t$. When $d(X^t, Y) < Z_t = n$ by parity $d(X^t, Y) \leq Z_t - 2$ and so we still have that $d(X^{t+1}, Y) \leq Z_{t+1}$. Because it reflects at n, the random walk is not a martingale but the modified walk $Y_t = Z_t + |\{s : s < t, Z_s = n\}|$ is a martingale. Then

$$E[Y_{T'}] = E[Y_0] = j$$

and so

$$E[\# \text{ of times Z hits } n] = j$$

and the expected number of moves between $n-1$ and n is $2j$. By excising moves above k and treating it as the boundary we similarly get

$$E[\# \text{ of times Z hits } k] \leq j$$

and so adding up all these steps we get

$$E[T] \leq E[T'] \leq 2jn \leq 2n^2.$$

It follows that $P[T \geq 4n^2] \leq \frac{1}{2}$ and so $P[T \geq 4rn^2] \leq (\frac{1}{2})^r$.

Exercise 11 *(Open Problem) For 2-SAT how can you sample the SAT assignments?*