STAT 206A: Gibbs Measures	Elchanan Mossel
Lecture 3	
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1 Ensembles of Factor Graphs

An ensemble of factor graphs is a family of randomly chosen factor graphs. In this lecture's notes we define two models of random factor graphs and discuss some properties related to the geometry and connectivity of these models.

Definition 1 $G_N(k, M)$ *is a random factor graph with* N *variable nodes and* M *factor nodes. Each factor node is connected to* k *nodes, which are chosen uniformly at random among all the* $\binom{N}{k}$ *possible sets of* k *different variable nodes.*

Note that, according to this definition, two factor nodes may have the same neighborhood, i.e., both can be connected to the same set of k variable nodes. On this model, the average degree of the variable nodes is clearly kM/N.

Definition 2 $G_N(k, \alpha)$ *is a random factor graph with* N *variable nodes. The factor nodes are obtained by the following procedure. For each set of* k *variable nodes, a factor node adjacent only to these* k *variable nodes is added with probability* $\alpha N / \binom{N}{k}$*.*

The average degree of the variable nodes in this model is αk. In the remaining of these notes, we concentrate on the cases where N and M goes to infinity, while $\alpha, M/N$ and k are kept fixed. It is important to remark that under these conditions both models may be often treated as equivalent where $\alpha = M/N$.

1.1 Degree Distribution

Definition 3 *For a factor graph* F*, let* λ_i *be the fraction of variable nodes with degree* i *and* P_i *be the fraction of factor nodes with degree* i*. Also, let* $\Lambda(x) = \sum_{i=0}^{M} \lambda_i x^i$ *and* $P(x) = \sum_{i=0}^{N} P_i x^i$*.*

It is easy to check that $\Lambda(1) = P(1) = 1$ and that the average degree of the variable nodes and the factor nodes are respectively given by $\Lambda'(1)$ and $P'(1)$. In the $G_N(k, \alpha)$ model, the probability that a randomly chosen node has a given degree is distributed according to $\mathrm{Bin}(\binom{N}{k}, \alpha N/\binom{N}{k})$. In the case where $N \to \infty$ and α is fixed, this distribution can be approximated by a Poisson with parameter αk.

Claim 4 *Given r fixed variable nodes of F, the degrees of these nodes are approximately independent from one another. Since each degree is distributed according to the Poisson distribution, the joint probability governing the degrees of these nodes can be obtained by a product of r independent Poisson probability function.*

A proof for this claim can be obtained using the total variation distance between two distributions.

Definition 5 *Let μ and ν be two probability distribution on a given space X. We define the total variation distance as*

$$|\mu - \nu|_{TV} = \frac{1}{2} \sum_{x \in X} |\mu(x) - \nu(x)| = \sup_{A \subseteq X} |\mu(A) - \nu(A)|.$$

Letting d_i and d_j be the degrees of two randomly chosen variable nodes i and j, respectively, for the case of $r = 2$, it suffices to show that

$$|(d_i, d_j) - d_i \times d_j|_{\mathrm{TV}} \leq O(1/N),$$

where (d_i, d_j) is the joint distribution and $d_i \times d_j$ is the distribution where the degrees are independent. In order to show that this is indeed the case, consider the following two distributions. Let $\beta = \alpha N/\binom{N}{k}$. In the first distribution μ we choose each clasue containing either i or j with probability β independently. In the second distribution ν we choose two collections of clauses. First we choose among all clauses containing i where each clause is chosen indepndently with probability β. Then we do the same for all clauses containing j. We may couple the two distributions by making exactly the same choices for all clauses that contains exactly one the variables i and j. Note that (d_i, d_j) is the joint degree distribution of i and j under the first distribution and $d_i \times d_j$ is the distribution of (i, j) in the first and second collections. Finally, unless a clauses containing i and j is chosen either in ν or μ the two degrees will be equal. For fixed α the μ and μ probability of choosing such clause is $O(1/N)$. The result follows.

1.2 Connected Components

Consider the construction of a random factor graph where the factor nodes are added one by one. In the beginning of this process, when few factor nodes have been added, the factor graph consists with high probability[1] of only small[2] connected components. After a given number of factor nodes are added to the graph, a relatively large connected component (of size $\Theta(N)$) arises with high probability. This connected component is commonly referred to as the giant connected component.

Theorem 6 (Giant Connected Component) *There is α_c such that for $\alpha < \alpha_c$ all connected components are small ($O(\log N)$) with high probability. For $\alpha > \alpha_c$, then there is $c(\alpha) > 0$ such that the size of the largest connected component is $c(\alpha)N$.*

Note that the connected components of the factor graph correspond to a dependency among the variables of the factorized probability distribution. When $\alpha < \alpha_c$, each variable is then only dependent on a small ($O(\log N)$) subset of variables. When $\alpha > \alpha_c$, there is a set of $\Theta(N)$ variables dependent on one another.

Claim 7 *Let F' be a fixed factor graph on V vertices and F factor nodes. Let $Z_{F'}$ be the number of occurrences of F' in $G_N(k, \alpha)$. Then $\mathbf{E}Z_{F'} \sim N^{V-(k-1)F}$.*

Proof: We can obtain $\mathbf{E}Z_{F'}$ by

$$\mathbf{E}Z_{F'} = \binom{N}{V} \left(N^{-k+1}\right)^F \left(1 - N^{-k+1}\right)^{\binom{V}{2}-F}.$$

The last term of this equation can be considered constant, so we have

$$\mathbf{E}Z_{F'} = \binom{N}{V} \left(N^{-k+1}\right)^F \approx N^{V-(k-1)F}.$$

□

Notice that we have $\mathbf{E}Z_{F'} \simeq \Theta(N)$ when F' is a tree, $\mathbf{E}Z_{F'} \simeq \Theta(1)$ when F' has one edge more than a tree, and $\mathbf{E}Z_{F'} \simeq O(1/N)$ when F' has at least two edges more than a tree.

We are now going to give a heuristic argument for the size of the largest component.

Question 8 *What is the probability x that a randomly chosen variable node belongs to a small connected component?*

Claim 9

$$x = \mathbf{E}_\ell[x^{\ell(k-1)}],$$

where ℓ is a random variable with a Poisson distribution.

[1]We say that an event occurs with high probability when it occurs with probability going to 1 as $N \to \infty$

[2]When referring to the size of connected components, we employ the word small to denote connected components of size $O(\log N)$

Let i be a randomly chosen variable node. The number of factor nodes adjacent to i is given by the random variable ℓ. Consequently, the number of variable nodes having a common factor node neighbor with i is $\ell(k-1)$. So, this claim means that the probability that i belongs to a small connected component is equal to the probability that all these $\ell(k-1)$ variable nodes also belong to a small connected component. The equation for x yields

$$x = e^{-\alpha k} \sum_{r=0}^{\infty} \frac{(\alpha k)^r}{r!} x^{r(k-1)} = e^{-\alpha k} e^{\alpha k x^{k-1}} = \exp(-\alpha k(1 - x^{k-1})).$$

Notice that regarding the variable nodes having a common factor node with i, this argument presumes that their degrees are all independent and that the probability that each of these variable nodes belong to a small connected component is also x.

The calculations above can also be obtained using branching processes. Note that the size of the connected component of a given variable node i is equivalent to the number of variable nodes reached during a branching process starting at i. We can then discover whether i belongs to a small connected component by checking if the expected number of variable nodes the branching process reach in each step is less than 1. The value of α for which the branching process is expected to reach exactly 1 variable node marks a phase transition for the behavior of the connected components of a random factor graph. More precisely, we state the following theorem.

Theorem 10 *If* $\alpha < \frac{1}{k(k-1)}$*, then* $\exp(-\alpha k(1 - x^{k-1}))$ *has a unique solution* $x = 1$ *and with high probability all connected components are of size* $O(\log N)$*. On the other hand, if* $\alpha > \frac{1}{k(k-1)}$*, then there is a solution* $0 < x^* < 1$ *and the size of the giant connected component is given by* $(1 - x^*)N(1 + o(1))$*.*

We conclude deriving the degree distribution of a variable node from the point of view of its incident edges. That is, we now pick a randomly chosen edge of the factor graph and want to obtain the probability distribution for the degree of the variable node incident to this edge.

Question 11 *Consider a factor graph where the degrees of the variable nodes are distributed as* $\Lambda(x) = \sum_{i=0}^{M} \lambda_i x^i$ *and the degrees of the factor nodes are distributed as* $P(x) = \sum_{i=0}^{N} P_i x^i$*. Given an edge chosen uniformly at random, what is the probability that the variable node incident to this edge has degree* ℓ*?*

This probability is proportional to the fraction of edges incident to degree-ℓ variable nodes, that is, it is equal to

$$\frac{\ell \lambda_\ell}{\sum_{\ell=0}^{M} \ell \lambda_\ell}.$$