Lecture 12

Lecture date: Oct 5 *Scribe: Guy Bresler*

In the previous lecture we derived an expression for the expected number of codewords having weight $w = \rho N$ for a general $\mathrm{LDPC}_N(\Lambda, P)$ code:

$$\bar{W}(w) = \sum_{E=0}^{F} \frac{E!(F-E)!}{F!} \operatorname{coeff}\left[\prod_{l=1}^{l_{\max}} (1+xy^l)^{N\Lambda_l}, x^w y^E\right] \operatorname{coeff}\left[\prod_{k=2}^{k_{\max}} q_k(z)^{MP_k}, z^E\right].$$

Recall that E denotes the number of red edges, and F denotes the total number of edges. Then $E = \xi F = \xi N \Lambda'(1)$ with ξ defined appropriately.

In this lecture we use the saddle point method to derive an approximation to $\operatorname{coeff}\left[\prod_{k=2}^{k_{\max}} q_k(z)^{MP_k}, z^E\right]$, where $q_k(z) = \frac{1}{2}\left((1+z)^n + (1-z)^n\right) = \sum_{r \text{ even}} \binom{k}{r} z^r$. This coefficient counts the number of ways to choose which edges incident to the check nodes are colored red. The other coeff[...] can be dealt with similarly.

1 Saddle Point Approximation

First, by Cauchy's Theorem,

$$\operatorname{coeff}\left[\prod_{k=2}^{k_{\max}} q_k(z)^{MP_k}, z^E\right] = \oint \frac{1}{z^{E+1}} \prod_{k=2}^{k_{\max}} q_k(z)^{MP_k} \frac{dz}{2\pi i} = \oint \frac{F(z)^N}{z} \frac{dz}{2\pi i},$$

where the integral runs over any simple closed path containing the origin, and we define

$$F(z) = \frac{1}{z^{\Lambda'(1)\xi}} \prod_{k=2}^{k_{\max}} q_k(z)^{\Lambda'(1)P_k/P'(1)}.$$

Claim 1 *Let C_r be the simple closed path of radius r in the counter-clockwise direction (i.e. the path given by $z = re^{it}$ for $-\pi \le t < \pi$). The maximum of $|F(z)|$ on C_r is obtained at $z = r$ and at $z = -r$.*

Proof: Observe that $|q_k(z)|$ obtains its maximum at $z = \pm r$. □

Exercise 2 *(1.5 points) $F : R^+ \to R^+$ has a unique minimum z_* given by solving*

$$\xi = \sum_{k=2}^{k_{max}} \rho_k z \frac{(1+z)^{k-1} - (1-z)^{k-1}}{1+z)^k + (1-z)^k}.$$

For the exercise, recall the definition $\rho_k = kP_k/P'(1)$.

Claim 3 *The following inequality holds:*

$$\oint_{C_{z_*}} \frac{F(z)^N}{z} \frac{dz}{2\pi i} \geq \Theta\left(\frac{F(z_*)^N}{\sqrt{N}}\right).$$

Proof: We will prove the claim for the case were z_* is the unique maximum of $F(z)$ on C_{z_*}. The case where $-z_*$ is also a maximum is treated similarly.

Let $\epsilon = N^{-0.4}$ (can choose $\epsilon = N^{-\delta}$ for $1/3 < \delta < 1/2$). Choose $g(z)$ so that $e^{g(z)} = F(z)$ and $g(z_*)$ is real. Then $g'(z_*) = 0$ and $g''(z_*) > 0$.

Exercise 4 *(1 point) Show that $g''(z_*) > 0$ (strictly greater than zero).*

Note that we may write

$$\oint_{C_{z_*}} \frac{F(z)^N}{z} \frac{dz}{2\pi i} = \int_{-\epsilon}^{\epsilon} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt + \int_{-\pi}^{-\epsilon} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt + \int_{\epsilon}^{\pi} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt.$$

We first find the contribution from the first integral and then show that the other two integrals may be neglected.

$$\begin{aligned}
\int_{-\epsilon}^{\epsilon} \frac{F(z_* e^{it})^N}{z_* e^{it}} &\overset{(a)}{=} (1 - O(\epsilon)) \int_{-\epsilon}^{\epsilon} e^{N\left[g(z_*) + \frac{g''(z_*)}{2}(1-e^{it})^2 (z_*)^2 + O(t^3)\right]} \frac{dt}{z_*} \\
&= (1 - O(\epsilon)) \int_{-\epsilon}^{\epsilon} F(z_*)^N \times \\
&\quad \exp\left[N \frac{g''(z_*)}{2}\left(-\sin^2 t + (1-\cos t)^2 + 2i \sin t (1 - \cos t)\right) + NO(t^3)\right] \frac{dt}{z_*} \\
&\overset{(b)}{=} (1 - O(\epsilon)) \int_{-N^{-0.4}}^{N^{-0.4}} F(z_*)^N \exp\left[N \frac{g''(z_*)}{2}\left(-t^2 + O(t^4) + O(t^3)\right)\right] \frac{dt}{z_* e^{it}} \\
&\overset{(c)}{=} (1 - O(\epsilon)) \frac{F(z_*)^N}{\sqrt{N}} \int_{-N^{0.1}}^{N^{0.1}} \exp\left[\frac{g''(z_*)}{2}\left(-t^2 + O\left(\frac{t^3}{\sqrt{N}}\right)\right)\right] \frac{dt}{z_*} \\
&\overset{(d)}{=} \Theta\left(\frac{F(z_*)^N}{\sqrt{N}}\right),
\end{aligned}$$

with the following steps at each labeled equality:
(a) Using the local expansion of g around z_*
(b) The asymptotic notation is used for $t \to 0$. For example, $(1 - \cos t)^2 = O(t^4)$.

(c) Change of variable: $t' = \sqrt{N}t$.
(d) The integral is Gaussian (contributing a constant).

Next, we show that the integral $\int_{\epsilon}^{\pi} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt$ may be neglected. The integral $\int_{-\pi}^{-\epsilon} \frac{F(z_* e^{it})^N}{z_* e^{it}}$ is treated similarly.

In order to bound the integral we bound separately

$$\int_{[0,\eta]\setminus[0,\epsilon]} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt$$

and

$$\int_{[0,\pi]\setminus[0,\eta]} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt,$$

where η is some small constant (independent of n) which is chosen so that: for $t \in ([-\eta, \eta] \setminus [-\epsilon, \epsilon])$ it holds that

$$\begin{aligned} |e^{g(z)}| &= \left| \exp\left[g(z_*) + \frac{g''(z_*)}{2}(1-e^{it})^2 + \cdots \right] \right| \\ &\leq \left| \exp\left[g(z_*) - \frac{1}{4N^{0.8}} g''(z_*) \right] \right|, \end{aligned}$$

so

$$\begin{aligned} \left| \int_{[0,\eta]\setminus[0,\epsilon]} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt \right| &\leq \int_{[0,\eta]\setminus[0,\epsilon]} \left| e^{N\left[g(z_*) + \frac{g''(z_*)}{2}(1-e^{it})^2 (z_*)^2 + \cdots \right]} \right| \frac{dt}{z_* e^{it}} \\ &= O\left(F(z_*)^N \exp\left[-\frac{N^{0.2} g''(z_*)}{4} \right] \right). \end{aligned}$$

For $t \in ([-\pi, \pi] \setminus [-\eta, \eta])$ it holds that $F(z_* e^{it}) \leq (1 - \Omega(1)) F(z_*)$ (because $F(z_*)$ is a unique maximum on $[-\pi, \pi]$), so

$$\left| \int_{[-\pi/2, \pi/2)\setminus[-\eta,\eta]} \frac{F(z_* e^{it})^N}{z_* e^{it}} dt \right| = O\left((1 - \Omega(1))^N F(z_*)^N \right).$$

□

Using the saddle point method to approximate the other coeff[...] expression gives the following:

$$\begin{aligned} \varphi(\rho) &= \sup_{\xi \in \{0,1\}} \inf_{\substack{x \geq 0 \\ y \geq 0 \\ z \geq 0}} \left\{ -\Lambda'(1) H(\xi) - \rho \log x - \Lambda'(1) \xi \log(yz) + \sum_{l=2}^{l_{\max}} \Lambda_l \log(1 + xy^l) + \frac{\Lambda'(1)}{P'(1)} \sum_{k=2}^{k_{\max}} P_k \log q_k(z) \right\} \\ &= -\rho \log x - \Lambda'(1) \log(1 + yz) + \sum_{l=1}^{l_{\max}} \Lambda_l (1 + xy^l) + \frac{\Lambda'(1)}{P'(1)} \sum_{k=2}^{k_{\max}} P_k \log q_k(z), \end{aligned}$$

where

$$\rho = \sum_{l=1}^{l_{\max}} \Lambda_l \frac{xy^l}{1+xy^l} \qquad\qquad y = \frac{\sum_{k=2}^{k_{\max}} \rho_k P_k^-(z)}{\sum_{k=2}^{k_{\max}} \rho_k P_k^+(z)}$$

$$z = \frac{\sum_{l=1}^{l_{\max}} \lambda_l xy^{l-1}/(1+xy^l)}{\sum_{l=1}^{l_{\max}} \lambda_l/(1+xy^l)} \qquad P_k^{\pm}(z) = \frac{(1+z)^{k-1} \pm (1-z)^{k-1}}{(1+z)^k + (1-z)^k}.$$