1 Ising Model on Trees

In this lecture we examine when uniqueness holds for the Ising Model. The first section gives an exact result for trees; the second section gives a sketch of a general bound comparing graphs to trees. In the first section, we consider l-level d-ary trees T_l (i.e. each node has d children). Recall that the probability of a spin assignment σ is

$$\mathbf{P}(\sigma) = \frac{1}{Z} \exp\left(\beta \sum_{i \sim j} \sigma_i \sigma_j + h \sum_i \sigma_i\right).$$

Definition 1 $R_v = \mathbf{P}(\sigma_v = -)/\mathbf{P}(\sigma_v = +)$

The next claim gives a recursive expression for the ratio R_v.

Claim 2 *Suppose v is the root, and $v_1, \ldots, v_d$ are the children of v. Then, for all boundary conditions,*

$$R_v = e^{-2h} \prod_{i=1}^{d} F(R_{v_i}) \qquad F(R) = \frac{e^{2\beta} R + 1}{e^{2\beta} + R},$$

where R_{v_i} is the above-defined ratio for the subtree rooted at v_i.

Proof:
step 1: Let $\mathbf{P}'(\sigma) = Z^{-1} \exp\left(\beta \sum_{i \sim j} \sigma_i \sigma_j + h \sum_{j \neq v} \sigma_j\right)$. Then

$$R_v = \frac{\mathbf{P}(\sigma_v = -)}{\mathbf{P}(\sigma_v = +)} = \frac{\mathbf{P}'(\sigma_v = -)}{\mathbf{P}'(\sigma_v = +)} e^{-2h}.$$

step 2: Define $\mathbf{P}'_i$ on the graph resulting from removing all subtrees of children of v from the original tree, except v_i and its subtree. Then

$$\frac{\mathbf{P}'(\sigma_v = -)}{\mathbf{P}'(\sigma_v = +)} = \prod_{i=1}^{d} \frac{\mathbf{P}'_i(\sigma_v = -)}{\mathbf{P}'_i(\sigma_v = +)}.$$

step 3: Define $\mathbf{P}''_i$ on the subtree rooted at v_i. Noting that the difference between $\mathbf{P}'_i$ and $\mathbf{P}''_i$ is only due to the potential on the edge $v \sim v_i$, we have

$$\frac{\mathbf{P}'_i(\sigma_v = -)}{\mathbf{P}'_i(\sigma_v = +)} = \frac{e^{\beta}\mathbf{P}''_i(\sigma_{v_i} = -) + e^{-\beta}\mathbf{P}''_i(\sigma_{v_i} = +)}{e^{-\beta}\mathbf{P}''_i(\sigma_{v_i} = -) + e^{\beta}\mathbf{P}''_i(\sigma_{v_i} = +)}$$
$$= \frac{e^{2\beta}R_{v_i} + 1}{R_{v_i} + e^{2\beta}} = F(R_{v_i}).$$

Note that this holds for all boundary conditions. Combining the expressions from each step gives the desired result. $\square$

Observe that $F(\infty) = e^{2\beta}$ and $F(0) = e^{-2\beta}$. ∞ corresponds to boundary conditions all $(-)$ and 0 corresponds to boundary conditions all $(+)$.

Claim 3 *Suppose $\beta \geq 0$. Then the boundary condition at level l maximizing $\mathbf{P}(\sigma(v) = +|\sigma_l)$ is given by $\sigma_l = +$.*

Proof: Follows from previous claim and next exercise. $\square$

Exercise 4 (1 point) *Prove that when $\beta \geq 0$, F is increasing, and when $\beta < 0$ then F is decreasing.*

Claim 5 *For $\{T_l\}$, the set of d-regular trees with l levels, uniqueness holds if and only if*

$$\lim_{l\to\infty} \mathbf{P}_l(\sigma(\mathit{root}) = +|\sigma_l = +) - \mathbf{P}_l(\sigma(\mathit{root}) = +|\sigma_l = -) = 0.$$

Proof:($\Leftarrow$) Clearly, by the definition of uniqueness, if the limit is not equal to zero, then uniqueness doesn't hold.
($\Rightarrow$) First, by Claim 3,

$$\lim_{l\to\infty} \sup_{\tau,\tau'} \mathbf{P}_l(\sigma(\text{root}) = +|\sigma_l = \tau) - \mathbf{P}_l(\sigma(\text{root}) = +|\sigma_l = \tau') = 0.$$

It remains to show that for all k,

$$\lim_{l\to\infty} \sup_{\tau,\tau'} \mathbf{P}_l\left(\sigma\left(B(r,k)\right) = \cdot\,|\sigma_l = \tau\right) - \mathbf{P}_l\left(\sigma\left(B(r,k)\right) = \cdot\,|\sigma_l = \tau'\right) = 0,$$

which is left as an exercise. $\square$

Exercise 6 (2 points) *Complete the proof.*

Claim 7 *Suppose $\beta \geq 0$, and let $g(x) = \log(e^{2\beta+x}+1) - \log(e^{2\beta}+e^x)$. Then uniqueness holds if and only if the equation*

$$x = -2h + dg(x)$$

has a unique fixed point.

Proof: Let $x_l = \log R_v(T_l)$. The recursion $x_{l+1} = -2h + dg(x_l)$ follows from Claim 2. The condition in Claim 5 holds if and only if $x = -2h + dg(x)$ has a unique fixed point since g is monotone (see below).□

Claim 8 *g is: 1) bounded, 2) increasing, 3) odd, 4)the max of $g'(x)$ is obtained at $x = 0$, 5) g' is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$.*

Using these properties, the next claim summarizes when uniqueness holds.

Claim 9 *There exists a $\beta_c(d)$ such that if $\beta \leq \beta_(d)$, then the recursion has a unique fixed point for all h which implies uniqueness for all h. If $\beta > \beta_c(d)$, then there exists $h(\beta) > 0$ such that $|h| < h(\beta) \Leftrightarrow$ non-uniqueness. Also in the later case 0 is repulsive, and the other two fixed points are attractors.*

Exercise 10 *What happens in this situation where $h = h(\beta)$? What are the intervals of attraction for each fixed point? (1 point)*

Finally from the properties of g it follows that:

Claim 11 *The critical β, $\beta_c(d)$, satisfies the following equation:*

$$dg'(0) = d\frac{e^{2\beta}-1}{e^{2\beta}+1} = 1.$$

2 Ising Model on General Graphs

In this section we sketch a procedure that was invented by D. Weitz for calculating R_v for general graphs by constructing a special tree corresponding to the original graph and using the recursion from the previous section on this tree. We use the following model.

$$P(\sigma) = \frac{1}{Z}\exp(\sum_{i\sim j}\beta_{ij}\sigma_{ij}).$$

We would ordinarily have a $\sum h_i \sigma_i$ term, but we leave it out for simplicity. Fix an ordering of the neighbors of v, i.e. let v have neighbors $u_1, \ldots, u_d$ in G. Form the graph G' from G by removing v and putting a new vertex v_i adjacent to u_i for $1 \le i \le d$. Then

$$\begin{aligned}
R_v &= \frac{\mathbf{P}(\sigma_v = -)}{\mathbf{P}(\sigma_v = +)} \\
&= \frac{\mathbf{P}_{G'}(\sigma_{v_1} = \cdots = \sigma_{v_d} = -)}{\mathbf{P}_{G'}(\sigma_{v_1} = \cdots = \sigma_{v_d} = +)} \\
&= \prod_{i=1}^{d} \frac{\mathbf{P}_{G'}(\sigma_{v_i} = - | \sigma_{v_1} \ldots \sigma_{v_{i-1}} = +, \sigma_{v_{i+1}} \ldots \sigma_{v_d} = -)}{\mathbf{P}_{G'}(\sigma_{v_i} = + | \sigma_{v_1} \ldots \sigma_{v_{i-1}} = +, \sigma_{v_{i+1}} \ldots \sigma_{v_d} = -)} \\
&= \prod_{i=1}^{d} R_{v_i}^{\tau_i},
\end{aligned}$$

where $R_{v_i}^{\tau_i} = F(R_{u_i}^{\tau_i})$ using the same argument as in step 3 of Claim 2, and

$$R_{u_i}^{\tau_i} = \frac{\mathbf{P}_{G'}(\sigma_{u_i} = - | \sigma_{v_1} \cdots \sigma_{v_{i-1}} = -, \sigma_{v_{i+1}} \cdots \sigma_{v_d} = +)}{\mathbf{P}_{G'}(\sigma_{u_i} = + | \sigma_{v_1} \cdots \sigma_{v_{i-1}} = -, \sigma_{v_{i+1}} \cdots \sigma_{v_d} = +)}.$$

In this process we have removed all edges from v and made v to be a part of a tree. Repeating this process recursively on $\{u_j\}$ gives an expression for R_v. By the construction of the recursion, this expression for R_v is the same as that attained for a particular tree (defined next) using the results for trees.

Let $T_v(G)$ be the tree of self-avoiding walks starting at v. Whenever a loop is closed from $w \to w'$, fix $\sigma_{w'} = +$ if $w' < w$, and $\sigma_{w'} = -$ if $w' > w$ (according to the ordering chosen initially). Thus some of the leaves of $T_v(G)$ are fixed, while the remaining leaves and nodes are not. It holds that

$$\frac{\mathbf{P}_G(\sigma_v = -)}{\mathbf{P}_G(\sigma_v = +)} = \frac{\mathbf{P}_{T_v(G)}(\sigma_v = -)}{\mathbf{P}_{T_v(G)}(\sigma_v = +)}$$

for all boundary conditions.

References

[1] D. Weitz, "Counting Independent Sets up to the Tree Threshold," *Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing*, 2006.