STAT 206A: Gibbs Measures Fall 2006

Lecture 16

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Recall that a distribution tree factorizes according to some factor graph $([N], [M], \{\partial a : 1 \leq a \leq M\}))$,

$$P(x) = \frac{1}{Z} \prod_{a=1}^{M} \psi_a(x_{\partial a}).$$

Definition 1 *A family of factorized distributions is $\{P_n\}$, where each P_n factorizes.*

Example 2 *The free 2-dimensional β Ising model: G_n are graphs ($n \times n$ grids) and*

$$P_n(\sigma) = \frac{1}{Z} \exp\left(\beta \sum_{\substack{i,j \in G_n \\ i \sim j}} \sigma_i \sigma_j\right).$$

Example 3 *The 2-dimensional β Ising model: Let $\partial_V G_n$ denote the vertex boundary and let $\tau \in \{-1, 1\}^{\partial_V G_n}$. Then $(P_n^\tau)_{1 \leq n < \infty}$ are given by*

$$P_n^\tau(\sigma) = \frac{1}{Z} \exp\left(\beta \sum_{\substack{i,j \in G_n \\ i \sim j}} \sigma_i \sigma_j\right) \exp\left(\sum_{i \in \partial_V G_n} \tau_i \sigma_i\right).$$

Note that the same procedure can be applied when you have an infinite factor graph G_V, factor node potentials (ψ) and a sequence $G_n \uparrow G$.

Example 4 *3-Regular Graphs: Consider the collection of all 3-regular graphs, $\{G_n^J\}$, where J is the edge function on G. Let*

$$P_n^J(x) = \frac{1}{Z} \prod_{i \sim j} \exp\left(\beta J_{i,j} x_i x_j\right).$$

Definition 5 *Let G be a factor graph and v a variable node in G. Let $B(v,l) = \{w : d(v,w) \leq l\}$ and $S(v,l) = \{w : d(v,w) = l\}$.*

Definition 6 *Suppose P_n factorizes according to G_n and let A and B be disjoint subsets of vertices of G_n. We define the following quantities:*

$$P_{n,A}^{\sigma_B}(\sigma_A) = P_n(\sigma|_A = \sigma_A \mid \sigma_B)$$

$$[A : B]_{P_n} = (P_{n,A}^{\sigma_B})_{all\ \sigma_B}$$

$$\operatorname{diam}[A : B]_{P_n} := \sup_{\sigma_B, \sigma'_B} d_{\mathrm{TV}}\left(P_{n,A}^{\sigma_B}, P_{n,A}^{\sigma'_B}\right),$$

where d_{TV} is the total variation distance.

Definition 7 *Let P_n be a family of factorized distributions. We say that uniqueness holds if $\forall l\ \forall \epsilon > 0\ \exists r > \ell$ such that $\forall n\ \forall v$*

$$\operatorname{diam}\left[B(v,l) : S(v,r)\right]_{P_n} < \epsilon.$$

That is, uniqueness means that the probability distribution on the l-ball isn't affected by what we see on the boundary of the r ball (where $r > l$).

A variant of uniqueness is exponential uniqueness. The definition of exponential uniqueness is essentially the same as for uniqueness except that we only consider ϵ of the form $\epsilon = (1-\eta)^{r-l}$.

As a final note to this discussion of uniqueness, we mention the connection between Definition 7 and the notion of uniqueness for infinite graphs: Given ψ (the factor node potentials), there exists a unique measure satisfying the Markov Property. This notion of uniqueness on infinite graphs is equivalent to uniqueness given by Definition 7 for all finite subgraphs.

Definition 8

$$\langle A : B \rangle = \sup_{f,g} \{\mathbf{E}_n[fg] : f \, depends\ on\ \sigma_A, g\ depends\ only\ on\ \sigma_B, \mathbf{E}[f] = \mathbf{E}[g] = 0, |f|_\infty < 1, |g|_{\infty < 1}\}$$

Definition 9 *We say that non-reconstruction holds if $\forall l\ \forall \epsilon > 0\ \exists r > \ell$ such that $\forall n\ \forall v$ $\langle B(v,l), S(v,r) \rangle \leq \epsilon$.*

Informally, in determining non-reconstruction, we want to know if in the factorized measure and with no conditioning we can get any information about the l-ball from the boundary of the r ball. If not, we call it non-reconstruction.

Proposition 1: Uniqueness implies non-reconstruction.

Proof: We want to compute $\mathbf{E}[fg]$. Then,

$$\mathbf{E}[fg|\sigma_{S(v,r)} = \overline{\sigma}] - \mathbf{E}[f]\mathbf{E}[g] = g(\overline{\sigma})\mathbf{E}[f|\sigma_{S(v,r)} = \overline{\sigma}]$$

because $\mathbf{E}[f] = \mathbf{E}[g] = 0$. From uniqueness,

$$d_{\mathrm{TV}}\left(P^{\sigma_{S(v,r)}}_{n,B(v,l)}, P_{n,B(v,l)}\right) \leq \epsilon$$

Thus,

$$\left|\mathbf{E}[f] - \mathbf{E}[f|\sigma_{S(v,r)} = \overline{\sigma}]\right| \leq 2\epsilon$$

and therefore:

$$\mathbf{E}[fg] \leq 2\epsilon E[|g|] \leq 2\epsilon.$$

□

Proposition 2: Non-reconstruction implies $\forall\epsilon > 0$ $\exists r$ such that $\forall n$, $\forall f, g$ with $|f|_\infty \leq 1$, $|g|_\infty \leq 1$, $\forall u, v$ such that $d(u,v) \geq r$

$$\mathbf{Cov}[f(\sigma_u), g(\sigma_v)] \leq \epsilon.$$

One dimensional systems. Here we show that for one dimensional systems we always have exponential uniqueness.

Claim 3: Consider a family $\{P_n\}$ where P_n factorizes over G_n. Assume

1. $\exists\eta > 0$ such that $\forall\psi$, $\eta \leq \psi \leq \frac{1}{\eta}$
2. $\exists D < \infty$ such that for every connected set of variable nodes, S, it holds that $|\partial_0 S| \leq D$, where $\partial_0 S = \{w \notin S : \exists v \in S, \partial_a : v, w \in \partial_a\}$

Then P_n has exponential uniqueness.

Proof: Let $P^{\sigma_1}, P^{\sigma_2} \in [B(v,l), S(v, l+r)]$. We'll show $d_{\mathrm{TV}}(P^{\sigma_1}, P^{\sigma_2}) \leq (1-\epsilon)^r$ by induction on r.

The case $r = 0$ is trivial. For $r = 1$, consider two probability measures, $Q^{\sigma_1}, Q^{\sigma_2} \in [B(v, l + (r-1), S(v, l+r)]$.

The boundedness of the potential coupled with the fact that the set $B(v, l + (r-1))$ has at most D neighbors implies that $\frac{dQ^{\sigma_1}}{dQ^{\sigma_2}} \in [\eta^{2D}, \eta^{-2D}]$. Therefore:

$$d_{\mathrm{TV}}(Q^{\sigma_1}, Q^{\sigma_2}) \leq 1 - \epsilon$$

where $\epsilon = \epsilon(D, \eta) > 0$.

This implies in particular that for $R^{\sigma_1}, R^{\sigma_2} \in [S(v, l + (r-1), S(v, l+r)]$ we have $d_{\mathrm{TV}}(R^{\sigma_1}, R^{\sigma_2}) \leq 1 - \epsilon$.

Now we consider each of the measures $P^{\sigma_1}, P^{\sigma_2}$ by first conditioning on the values in $S(v, l + (r-1))$ – we then obtain that:

$$P^{\sigma_1} - P^{\sigma_2} = (1-\epsilon) \sum l_i \left(P^{i,\tau_i} - P^{i,\overline{\tau_i}} \right)$$

where $l_i \geq 0$, $\sum l_i = 1$ and $P^{i,\tau_i}, P^{i,\overline{\tau_i}} \in [B(v,l) : S(v, l+r-1)]$. By the inductive hypothesis we have

$$d_{\mathrm{TV}}(P^{i,\tau_i}, P^{i,\overline{\tau_i}}) \leq (1-\epsilon)^{r-1},$$

for all i. Summing over all i, we obtain

$$d_T V(P^{\sigma_1}, P^{\sigma_2}) \leq (1-\epsilon)^r$$

as needed. □