In the previous lecture we saw how to express the function $\varphi(\rho)$ which is the normalized log of the expected number of code words with relative weight $\rho$. This was given by the formula:

$$\varphi(\rho) = \sup_{\xi \in \{0, 1\}} \inf_{x \geq 0, y \geq 0, z \geq 0} \left\{-\Lambda'(1)H(\xi) - \rho \log x - \Lambda'(1)\xi \log(yz) + \sum_{l=2}^{l_{\text{max}}} \Lambda_l \log(1 + xy^l) + \frac{\Lambda'(1)}{P'(1)} \sum_{k=2}^{k_{\text{max}}} P_k \log q_k(z)\right\}$$

$$= -\rho \log x - \Lambda'(1) \log(1 + yz) + \sum_{l=2}^{l_{\text{max}}} \Lambda_l (1 + xy^l) + \frac{\Lambda'(1)}{P'(1)} \sum_{k=2}^{k_{\text{max}}} P_k \log q_k(z),$$

where

$$\rho = \sum_{l=1}^{l_{\text{max}}} \Lambda_l \frac{xy^l}{1 + xy^l}, \quad y = \frac{\sum_{k=2}^{k_{\text{max}}} \rho_k P_k^{-}(z)}{\sum_{k=2}^{k_{\text{max}}} \rho_k P_k^{+}(z)} \quad \text{and} \quad P_k^{\pm}(z) = \frac{(1 + z)^{k-1} \pm (1 - z)^{k-1}}{(1 + z)^k + (1 - z)^k}.$$
\[ z \sim \frac{\lambda_{\ell(\text{min})}^x y^{\ell(\text{min})-1}}{\sum \lambda_{\ell}} = \lambda_{\ell(\text{min})}^x y^{\ell(\text{min})-1}, \]

and

\[ \rho \sim \Lambda_{\ell(\text{min})}^x y^{\ell(\text{min})-1}. \]

From the corollary, it is clear that the short distance properties depend very strongly on the minimal possible variable degree. We will discuss the three cases: \( \ell(\text{min}) = 1 \), \( \ell(\text{min}) = 2 \) and \( \ell(\text{min}) \geq 3 \).

**\( \ell(\text{min}) = 1 \)**. In this case we obtain:

\[ y \sim \rho'(z), \quad z \sim \lambda_1 x, \quad \rho \sim \Lambda_1 xy \]

and therefore

**Corollary 2**

\[ \varphi(\rho) = -\frac{1}{2} \rho \log \rho + O(\rho) \]

and therefore for each \( \rho \) the expected number of code words of weight \( \rho \) is exponential in \( n \).

In fact one can obtain the fact that in the case \( \ell(\text{min}) = 1 \) there are many codewords of small weight also with high-probability observing that

**Claim 3** If \( \ell(\text{min}) = 1 \) then w.h.p. every codeword has \( \Omega(N) \) code-words at distance 2 from it.

The proof of the claim follows by observing that w.h.p there is a linear fraction of factor nodes connected to variable nodes of degree 1 only.

**\( \ell(\text{min}) = 2 \)**

**Claim 4** If \( \ell(\text{min}) = 2 \) then \( \varphi(\rho) \sim A\rho \) where

\[ A = \log \frac{P''(1)2\Lambda_2}{P'(1)\Lambda'(1)} \]

The proof of this claim is left as an exercise (1 point)
Claim 5  If $\ell(\text{min}) = 3$ then

$$\varphi(\rho) \sim \frac{\ell(\text{min}) - 2}{2} \rho \log(\rho/\Lambda_{\ell(\text{min})}).$$

The proof of this claim is left as an exercise (1 point).

Small linear distances and sub-linear distances  Using the previous two claims and a first moment argument we obtain:

Corollary 6  Consider LDPC with $\ell(\text{min}) \geq 3$ or $\ell(\text{min}) = 2$ and $A < 0$. Let $\rho^*$ be the first non-trivial zero of $\varphi$. Then for any open interval $(\rho_1, \rho_2) \subset [0, \rho^*)$ it holds that w.h.p there are no code words with weight in the interval $N(\rho_1, \rho_2)$.

Remark 7  Note that the claim above does not exclude the case of codewords of sub-linear weight. In fact,

- When $\ell(\text{min}) = 2$ a small (but positive) number of code-words of sub-linear weight exists with high probability.
- When $\ell(\text{min}) \geq 3$ w.h.p. there are no code-words of sub-linear weight. The proof of this fact is similar to expansion proofs we will see later.

0.1 Rate of LDPC codes

Recall that the rate of a linear code $C \subseteq F_2^n$ is given by $\log |C| / \log n$. We have seen that for any code with degree distribution $\Lambda', P'$ it holds that the rate $R$ of the code satisfies:

$$R \geq 1 - \frac{\Lambda'(1)}{P'(1)}.$$ 

We will now see that generally for LDPC codes, it holds that the rate is indeed given w.h.p by

$$R = 1 - \frac{\Lambda'(1)}{P'(1)}.$$ 

One way to find an upper bound on the rate is to upper bound the maximum value of $\varphi(\rho)$. It is natural to expect that the maximum is obtained at $\rho = 1/2$. 

12-3
Exercise 8  Find conditions on the degree distributions implying that the maximum of $\varphi$ is obtained at $\rho = 1/2$.

Claim 9  Suppose that the maximum of $\varphi$ is obtained at $\rho = 1/2$ and that $\delta > 0$ the w.h.p. it holds that
\[ \mathbb{R} \leq 1 - \frac{\Lambda'(1)}{P'(1)} + \delta. \]

Proof: Using a first moment argument it suffices to show that
\[ \varphi(1/2) \leq \log(2) \left( 1 - \frac{\Lambda'(1)}{P'(1)} \right). \]

Next one verifies that $\rho = 1/2$ correspond to $x = y = z = 1$ in the formula for $\varphi$. Plugging this into the formula then gives:
\[ \varphi(1/2) = \log(2) \left( 1 - \frac{\Lambda'(1)}{P'(1)} \right). \]

$\square$