Lecture 2

Lecture date: September 1, 2005 *Scribe: Richard Liang*

In this lecture, we discuss the space $L^2(\mu)$ of square-integrable functions on a probability space $(\Omega, \mathcal{F}, \mu)$, and introduce the notions of a *standard basis* for this space, and of *tensor products*.

1 $L^2(\mu)$

Define our probability space $(\Omega, \mathcal{F}, \mu)$. Let f be a random variable on this space, and we outline some notation.

$$
\begin{aligned}
\mathbf{E}f &= \int_\Omega f\,d\mu \\
|f|_2 &= \sqrt{\int_\Omega f^2 d\mu} \\
\mathbf{Var}\,f &= |f - \mathbf{E}f|_2^2 \\
\mathbf{Cov}(f,g) &= \mathbf{E}((f-\mathbf{E}f)(g-\mathbf{E}g)) \\
\langle f, g\rangle &= \mathbf{E}(f\overline{g})
\end{aligned}
$$

Definition 1 $L^2(\mu) = \{f : \Omega \to \mathbb{R}, |f|_2 < \infty\}$*; that is,* $L^2(\mu)$ *is the space of square-integrable random variables on the space* $(\Omega, \mathcal{F}, \mu)$*.*

Example 2 *Let* Ω *be a finite set, taking* $\mathcal{F} = 2^\Omega$*, and let the measure* μ *be non-degenerate, so that* $\mu(\omega) > 0$ *for all* $\omega \in \Omega$*. In this case,* $L^2(\mu)$ *is just* $(\mathbb{R}^{|\Omega|}, |\cdot|_2)$*, since any real-valued function on a finite set is square-summable.*

2 Standard bases

We next define the notion of a standard basis.

Definition 3 *A subset* $B = \{u_0, u_1, u_2, \ldots\}$ *of* $L^2(\mu)$ *is a* standard basis *of* $L^2(\mu)$ *if:*

- *the functions u_i are orthonormal;*
- *and $u_0 = 1$.*

If B is a standard basis, then for a separable space, by the orthonormality if $\dim(L^2(\mu)) = \infty$ and $f \in L^2(\mu)$, then

$$f = \sum_{i=0}^{\infty} \langle f, u_i \rangle \, u_i$$

(that is,

$$\left| f - \sum_{i=0}^{n} \langle f, u_i \rangle \, u_i \right|_2 \to 0$$

as $n \to \infty$). We will only consider separable spaces.

It is well known that if $L^2(\mu)$ is separable, then it has a standard basis.

Exercise 4 *(1 point) Prove that if Ω is finite, then it has a standard basis.*

The following basic properties of a standard basis will be useful.

Lemma 5 *Let $\{u_0, u_1, u_2, \ldots\}$ be a standard basis, and let $f, g \in L^2(\mu)$, so we can write $f = \sum_{i=0}^{\infty} a_i u_i$ and $g = \sum_{i=0}^{\infty} b_i u_i$. Then:*

- $\mathbf{E} f = a_0$
- $|f|_2^2 = \sum_i a_i^2$
- $\mathbf{Cov}(f, g) = \sum_{i=i}^{\infty} a_i b_i$
- $\mathbf{Var}(f) = \sum_{i=1}^{\infty} a_i^2$

Exercise 6 *(1 point) Prove Lemma 5 when Ω is finite.*

(As a hint, we will show how to prove the third point.

$$\begin{aligned} \mathbf{Cov}(f, g) &= \langle f - \mathbf{E} f, g - \mathbf{E} g \rangle \\ &= \left\langle \sum_{i=1}^{\infty} a_i u_i, \sum_{i=1}^{\infty} b_i u_i \right\rangle \text{ by the first point} \\ &= \sum_{i=1}^{\infty} a_i b_i \end{aligned}$$

as desired.)

Example 7 *(2 point space) In this simple example, we define $\Omega = \{-1, 1\}$, and for $-1 \leq \theta \leq 1$, define the measure on this space μ_θ by $\mu_\theta(\{1\}) = (1+\theta)/2$ and $\mu_\theta(\{-1\}) = (1-\theta)/2$. Thus, writing $X(\omega) = \omega$, $\mathbf{E}X = \theta$. (We will write, as shorthand for this space, $\{-1, 1\}_\theta$.)*

A standard basis for this space $\{u, v\}$ can be determined as follows. We must take $u = 1$. Then, we need $\mathbf{E}V = 0$ and $\mathbf{E}V^2 = 1$. Since on 2-point space any function must be linear, we can write $V(\omega) = a\omega + b$. $\mathbf{E}V = 0$, so

$$\begin{aligned} & a\theta + b = 0 \\ \Rightarrow & \, b = -a\theta \\ \Rightarrow & \, V(\omega) = a(\omega - \theta). \end{aligned}$$

Then, using the condition $\mathbf{E}V^2 = 1$, we get:

$$\begin{aligned} \mathbf{E}V^2 &= a^2\mathbf{E}((X-\theta)^2) \\ &= a^2\mathbf{E}(X^2 - 2\theta X + \theta^2) \\ &= a^2(1-\theta^2) \\ &= 1 \\ \Rightarrow a &= \frac{1}{\sqrt{1-\theta^2}}. \end{aligned}$$

So, we get a unique standard basis $\left\{1, \frac{X-\theta}{\sqrt{1-\theta^2}}\right\}$. In particular, if $\theta = 0$, the basis is just $\{1, X\}$.

Example 8 *A more interesting example is for the space $\Omega = [n]$, the cyclic group $\mathbb{Z}_n$, with the uniform measure. A* character *is a function $f : \mathbb{Z}_n \to \mathbb{C}$ which satisfies $f(x+y) = f(x)f(y)$.*

Claim: *The set of characters is a standard basis.*

In showing this, first we note two properties of characters: $f(0) = 1$, and $f(1)^n = 1$ since the group is cyclic. Therefore the characters are exactly, for $0 \leq j \leq n-1$,

$$f_j(k) = \exp\left(\frac{i2\pi jk}{n}\right).$$

This is a standard basis: $f_0 = 1$, $|f_j|_2^2 = 1$, and

$$\begin{aligned} \langle f_j, f_k \rangle &= \frac{1}{n}\sum_{r=0}^{n-1} f_j(r)\overline{f_k(r)} \\ &= \frac{1}{n}\sum_{r=0}^{n-1} f_{j-k}(r), \end{aligned}$$

and this sum is 0 *when* $j \neq k$*, because*

$$\begin{aligned}\sum_{r=0}^{n-1} f_l(r) &= \sum_{r=0}^{n-1} f_l(r+1) \text{ by cyclic-ness} \\ &= \exp\left(\frac{2\pi i l}{n}\right)\sum_{r=0}^{n-1} f_l(r)\end{aligned}$$

and so if $l \neq 0$*, the exponent on the right hand side is not* 0*, so the sum must be* 0*.*

3 Tensor products

For the probability spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$, we can consider the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$, where the measure $\mu_1 \times \mu_2$ is defined by, for $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, $\mu_1 \times \mu_2(A \times B) = \mu_1(A)\mu_2(B)$, and we deal with $L^2(\mu_1 \times \mu_2) = L^2(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$.

Definition 9 *Let* $f \in L^2(\mu_1)$ *and* $g \in L^2(\mu_2)$*. Define the* tensor product *of* f *and* g*,* $f \otimes g \in L^2(\mu_1 \times \mu_2)$*, by*

$$(f \otimes g)(x, y) = f(x)g(y).$$

An important property of tensor products is given by the following:

Proposition 10 *Let* $f_1, f_2 \in L^2(\mu_1)$ *and* $g_1, g_2 \in L^2(\mu_2)$*. Then*

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle .$$

In particular,

$$\mathbf{E}(f_1 \otimes g_1) = \mathbf{E} f_1 \mathbf{E} g_1.$$

The second part follows directly from the first part by setting $f_2 = g_2 = 1$. The first part comes from Fubini's theorem:

$$\begin{aligned}\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle &= \int (f_1 \otimes g_1)(x, y)(f_2 \otimes g_2)(x, y) d\mu_1(x) \times \mu_2(y) \\ &= \int f_1(x)g_1(y)f_2(x)g_2(y) d\mu_1 \times \mu_2 \\ \text{Fubini} \Rightarrow \quad &= \int f_1(x)f_2(x) d\mu_1 \int g_1(y)g_2(y) d\mu_2 \\ &= \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle .\end{aligned}$$

Dealing with tensor products requires many indices; to clean up the notation, we make the following convenient definition.

Definition 11 *I is a* multi-index *if $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$ (here $\mathbb{N}$ is taken to be the set of integers ≥ 0). The* weight *of I is $|I| = |\{k : i_k = 0\}|$.*

If we have bases for the spaces

$$\begin{aligned} U^1 &= \{u_0^1, u_1^1, \ldots, u_n^1\} \subset L^2(\mu_1) \\ U^2 &= \{u_0^2, u_1^2, \ldots, u_n^2\} \subset L^2(\mu_2) \\ &\vdots \\ U^m &= \{u_0^m, u_1^m, \ldots, u_n^m\} \subset L^2(\mu_m) \end{aligned}$$

we can define, for a multi-index I, $u_I = u_{i_1}^1 \otimes u_{i_2}^2 \otimes \cdots \otimes u_{i_m}^m$. Furthermore, we can define $U^1 \otimes U^2 \otimes \cdots \otimes U^m$ to be all elements of the form u_I for any multi-index I. The following lemma shows that this is a natural object to consider.

Lemma 12 *If U^i is a standard basis of $L^2(\mu_i)$ for all i from 1 to m, then $U^1 \otimes U^2 \otimes \cdots \otimes U^m$ is a standard basis of $L^2\left(\prod_{i=1}^m \mu_i\right)$.*

Proof: (For finite spaces) For convenience, we write U for $U^1 \otimes U^2 \otimes \cdots \otimes U^m$. First we see that $1 = 1 \otimes 1 \otimes \cdots \otimes 1$ is in U. Secondly,

$$\begin{aligned} \langle u_I, u_J \rangle &= \prod_{k=1}^m \left\langle u_{i_k}^k, u_{j_k}^k \right\rangle \\ &= \delta_{I,J} \text{ by the above proposition,} \end{aligned}$$

so the u_Is are orthonormal. Finally, checking that the number of these u_I equals the dimension of the whole space completes the argument (in the finite case). $\square$

Example 13 *Write $\{-1,1\}_\theta^m = \{-1,1\}_\theta^{\otimes m}$. We found above the standard bases for each of the component spaces. By the above lemma, then, a standard basis for the product space is given by, for all $S \subset [m]$,*

$$U_S(\omega_1, \omega_2, \ldots, \omega_m) = \prod_{i \in S} \frac{\omega_i - \theta}{\sqrt{1-\theta^2}}.$$

For example, if $\theta = 0$,

$$U_S(\omega_1, \omega_2, \ldots, \omega_m) = \prod_{i \in S} \omega_i.$$

Example 14 *A more interesting example of the kinds of spaces that arise as tensor products are finite Abelian groups. If $\Omega = G$ is a finite Abelian group under the uniform measure, we* *can write $G = G_1 \times G_2 \times \cdots \times G_k$, where the G_is are cyclic groups. We showed above that standard bases for such groups are given by their sets of characters; the above lemma thus tells us that a standard basis for G is given by the products $\psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_k(\omega_k)$, where ψ_i is a character for G_i.*

Definition 15 *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ be probability spaces, and let U^i be a standard basis for $L^2(\mu_i)$ for all i. For $S \subset [n]$, define $L^2_S(\prod_i \mu_i)$ to be the space spanned by u_I, where the multi-index I is such that:*

- *if $j \in S$, $i_j \neq 0$; and*
- *if $j \notin S$, $i_j = 0$.*

This is the space of all functions that are not constant in any of the indices in S. The next Lemma relates these sets to the spaces of all functions that *may vary* in any of the indices of S.

Lemma 16 *Let $L^2_{|S}(\prod_i \mu_i) \subset L^2(\prod_i \mu_i)$ be the space spanned by all functions of the form $f(x_1, \ldots, x_n) = g((x_i)_{i \in S})$; (Think of the $|$ as representing a restriction to S, as these are functions that are only allowed to depend on indices in S.) Then*

$$L^2_{|S}\left(\prod_i \mu_i\right) = \bigoplus_{T \subset S} L^2_T\left(\prod_i \mu_i\right)$$

(the $\oplus$ represents an orthogonal sum).

Proof: (For the finite case) The orthogonality of the spaces is clear by definition. If $T \subset S$, then clearly $L^2_T(\prod_i \mu_i) \subset L^2_{|S}(\prod_i \mu_i)$. Finally, counting dimensions completes the proof. □

Corollary 17 *The spaces L_S do not depend on the basis.*

This will be proven in the next lecture.