Lecture 16

Lecture date: Oct 20 *Scribe: Yun Long*

In this class, we finished the proof of the following theorem.

Theorem 1 *Let $q \in (2,3]$, $X_1, ..., X_n$ be independent random variables satisfying $\mathbf{E}[X_i] = 0$, $\mathbf{E}[X_i^2] = 1$, and $\mathbf{E}[|X_i|^q] \leq \beta$ for some constant $\beta > 0$. Let Q be a multi-linear polynomial of degree d that can be expressed as $Q(x_1, ..., x_n) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$. And the Fourier coefficients satisfies $\sum_{S \neq \emptyset} c_S^2 = 1$. Let $\Delta : \mathbb{R} \longrightarrow [0,1]$ be a non-decreasing function with $\Delta(0) = 0$, $\Delta(1) = 1$ and $\sup_x |\Delta^{(3)}(x)| = A < \infty$. Define $\Delta_r(x) = \Delta(\frac{x}{r})$. Then*

$$|\mathbf{E}[\Delta_r Q(X_1, ..., X_n)] - \mathbf{E}[\Delta_r Q(G_1, ..., G_n)]| \leq O_q(Ar^{-q}\beta^{d/q} \sum_i (\sum_{S: i \in S} c_S^2)^{q/2}) \tag{1}$$

Where $G_1, ..., G_n$ are independent and with standard normal distributions.

Proof: Split the left hand side of equation (1), we get,

$$\begin{aligned}
&|\mathbf{E}[\Delta_r Q(X_1, ..., X_n)] - \mathbf{E}[\Delta_r Q(G_1, ..., G_n)]| \\
&\leq |\mathbf{E}[\Delta_r Q(X_1, ..., X_n)] - \mathbf{E}[\Delta_r Q(G_1, X_2, ..., X_n)]| \\
&\quad + |\mathbf{E}[\Delta_r Q(G_1, X_2, ..., X_n)] - \mathbf{E}[\Delta_r Q(G_1, G_2, X_3, ..., X_n)]| \\
&\quad + ... \\
&\quad + |\mathbf{E}[\Delta_r Q(G_1, ..., G_{n-1}, X_n)] - \mathbf{E}[\Delta_r Q(G_1, ..., G_n)]|
\end{aligned}$$

We will prove the following claim:

$$\begin{aligned}
&|\mathbf{E}[\Delta_r Q(Z_1, ..., Z_{i-1}, X_i, Z_{i+1}, ..., Z_n)] - \mathbf{E}[\Delta_r Q(Z_1, ..., Z_{i-1}, G_i, Z_{i+1}, ..., Z_n)]| \\
&\qquad \leq O_q(Ar^{-q}\eta^{-d}(\sum_{S:\, i \in S} c_S^2)^{q/2})
\end{aligned} \tag{2}$$

Where $Z_j = X_j$, if $j > i$, $Z_j = G_j$, if $j < i$, and $\eta = \frac{\beta^{-1/q}}{2\sqrt{q-1}}$.

If this claim is proved, then summing (2) over all i lead to our desired upper bound (1).

Proof of the claim: Since Q is a multi-linear polynomial, we could write it as

$$Q(Z_1, ..., Z_{i-1}, W, Z_{i+1}, ..., Z_n) = R(Z) + S(Z) \cdot W$$

Here $Z = (Z_1, ..., Z_{i-1}, Z_{i+1}, ..., Z_n)$. Polynomial R and S satisfies:

$$R(Z_1, ..., Z_{i-1}, Z_{i+1}, ..., Z_n) = \sum_{S:\, i \notin S} c_S \prod_{j \in S} Z_j$$

$$S(Z_1, ..., Z_{i-1}, Z_{i+1}, ..., Z_n) = \sum_{S:\, i \in S} c_S \prod_{j \in S \setminus \{i\}} Z_j$$

Starting from the left hand side of (2). By the lemma we proved in last lecture:

$$\begin{aligned}\mathbf{E}[\Delta_r(R(Z) + S(Z)X_i)] - \mathbf{E}[\Delta_r(R(Z) + S(Z)G_i)] \\ \leq O_q(Ar^{-q})[\mathbf{E}(|S(Z)X_i|^q) + \mathbf{E}(|S(Z)G_i|^q)]\end{aligned}$$

By Lemma 7 of Lecture 13, all X_j and G_j are $(2, q, \eta)$ hyper-contractive with such defined η. By the claim we proved in last lecture,

$$O_q(Ar^{-q})[\mathbf{E}(|S(Z)X_i|^q) + \mathbf{E}(|S(Z)G_i|^q)] \leq O_q(Ar^{-q}\eta^{-d})[\|S(Z)X_i\|_2^q + \|S(Z)G_i\|_2^q]$$

Because $\|S(Z)X_i\|_2^2 = \|S(Z)G_i\|_2^2 = \sum_{S:\, i \in S} c_S^2$, we thus proved

$$\mathbf{E}[\Delta_r(R(Z) + S(Z)X_i)] - \mathbf{E}[\Delta_r(R(Z) + S(Z)G_i)] \leq O_q(Ar^{-q}\eta^{-d}) \sum_{S:\, i \in S} c_S^2$$

which is the claim we wanted. $\square$

Remark: If in addition, $I_i(Q) \leq \delta$ for all i. Since $\sum_{S:\, i \in S} c_S^2 = I_i(Q)$,

$$\begin{aligned}|\mathbf{E}[\Delta_r Q(X_1, ..., X_n)] - \mathbf{E}[\Delta_r Q(G_1, ..., G_n)]| &\leq O_q(Ar^{-q}\eta^{-d}(\sum_i I_i(Q))^{q/2}) \\ &\leq O_q(Ar^{-q}\eta^{-d})\delta^{q/2-1} \sum_i I_i \\ &\leq O_q(Ar^{-q}\eta^{-d})\delta^{q/2-1} d\end{aligned}$$

The last inequality uses the fact that $\sum_{S \neq \emptyset} c_S^2 = 1$, so,

$$\sum_i I_i = \sum_S c_S^2 |S| \leq d \sum_{S \neq \emptyset} c_S^2 = d$$