Lecture 14

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## 1 Hyper-Contraction for Sets of Random Variables

Let $\mathcal{X} = \{X_1, \ldots, X_k\}$, where $X_1, \ldots, X_k$ are random variables with of whose moments are finite. Denote by $\mathcal{P}_n(\mathcal{X})$ and $\mathcal{P}(\mathcal{X})$ the following sets

$$\mathcal{P}_n(\mathcal{X}) = \{\text{all polynomials of degree} \le n \text{ in variables from } \mathcal{X}\}$$
$$\mathcal{P}(\mathcal{X}) = \{\text{all polynomials in variables from } \mathcal{X}\}$$

Finally, let $T_\eta : \mathcal{P}(x) \to \mathcal{P}(x)$, where $\eta \in (0,1)$, be the linear operator satisfying the following property

$$T_\eta Y = \eta^n Y \text{ if } Y \in \mathcal{P}_n(\mathcal{X}) \cap \mathcal{P}_{n-1}^{\perp}(\mathcal{X})$$

**Definition 1** *Suppose* $1 \le p \le q < \infty$ *and* $0 < \eta < 1$. *We say that* $\mathcal{X}$ *is* $(p,q,\eta)$ *hypercontractive if for all polynomials* $Q \in \mathcal{P}(\mathcal{X})$ *the following is satisfied:*

$$\|T_\eta Q(\mathcal{X})\|_q \le \|Q(\mathcal{X})\|_p$$

**Remark 2** *The following are easy observations:*

- *If* $\{1, X_1, \ldots, X_k\}$ *is a standard basis for* $L^2(\mu)$*, for some measure* $\mu$*, then for every* $X$*:*
$$T_\eta X = \eta X + (1-\eta)\mathbb{E}[X]$$
*(to see why write* $X = (X - \mathbb{E}[X]) + \mathbb{E}[X]$ *and note that the two summands are orthogonal, the first summand is a degree* 1 *polynomial in* $X_1, \ldots, X_k$ *and the second summand a degree* 0 *polynomial)*

- *If* $\{1, X_1^1, \ldots, X_k^1\}$ *is a standard basis for* $L^2(\mu_1)$ *and* $\{1, X_1^2, \ldots, X_l^2\}$ *a standard basis for* $L^2(\mu_2)$*, for some measures* $\mu_1$ *and* $\mu_2$*, and* $\mathcal{X} = \{X_1^1, \ldots, X_k^1, X_1^2, \ldots, X_l^2\}$ *then:*
$$\mathcal{P}(\mathcal{X}) = \bigotimes_i L^2(\mu_i)$$
*and* $T_\eta$ *is the Bonami-Beckner operator*

**Claim 3** *If* $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ *and* $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ *are two* $(p,q,\eta)$ *hypercontractive sets of random variables that are independent, then so is* $\mathcal{X} \cup \mathcal{Y}$.

**Proof:** Let $Q$ be a polynomial in $\mathcal{X}$ and $\mathcal{Y}$. We can write $Q$ as follows:

$$Q(\mathcal{X} \cup \mathcal{Y}) = \sum_i R_i(\mathcal{X}) S_i(\mathcal{Y})$$

and therefore:

$$\begin{aligned}
\|T_\eta Q(\mathcal{X} \cup \mathcal{Y})\|_q &= \|\|T_\eta Q(\mathcal{X} \cup \mathcal{Y})\|_{L^q(\mathcal{Y})}\|_{L^q(\mathcal{X})} && \text{(Fubini's theorem)} \\
&= \left\| \left\| T_{\eta,\mathcal{Y}} \left( \sum_i T_{\eta,\mathcal{X}}(R_i(\mathcal{X}) S_i(\mathcal{Y})) \right) \right\|_{L^q(\mathcal{Y})} \right\|_{L^q(\mathcal{X})} \\
&\leq \left\| \left\| \sum_i T_{\eta,\mathcal{X}}(R_i(\mathcal{X}) S_i(\mathcal{Y})) \right\|_{L^p(\mathcal{Y})} \right\|_{L^q(\mathcal{X})} && \text{(hypercontractivity of } \mathcal{Y}) \\
&\leq \left\| \left\| T_{\eta,\mathcal{X}} \sum_i R_i(\mathcal{X}) S_i(\mathcal{Y}) \right\|_{L^q(\mathcal{X})} \right\|_{L^p(\mathcal{Y})} && \text{(generalized Minkowski inequality } (p \leq q)) \\
&\leq \left\| \left\| \sum_i R_i(\mathcal{X}) S_i(\mathcal{Y}) \right\|_{L^p(\mathcal{X})} \right\|_{L^p(\mathcal{Y})} && \text{(hypercontractivity of } \mathcal{X}) \\
&= \|Q(\mathcal{X} \cup \mathcal{Y})\|_p
\end{aligned}$$

□

**Claim 4** *Let $\mathcal{X}$ be a $(2, q, \eta)$ hypercontractive set of random variables and $Q$ a polynomial of degree $\leq d$ on $\mathcal{X}$. Then $\|Q(\mathcal{X})\|_q \leq \eta^{-d} \|Q(\mathcal{X})\|_2$.*

**Proof:** We distinguish the following cases:

- if $Q(\mathcal{X}) \in \mathcal{P}_d(\mathcal{X}) \cap \mathcal{P}_{d-1}^{\perp}(\mathcal{X})$, then $T_\eta Q(\mathcal{X}) = \eta^d Q(\mathcal{X})$ and so, by the hypercontractivity of $\mathcal{X}$, it follows that:

$$\|\eta^d Q(\mathcal{X})\|_q = \|T_\eta Q(\mathcal{X})\|_q \leq \|Q(\mathcal{X})\|_2$$

- in general, $Q = \sum_{i=0}^{d} Q_i(\mathcal{X})$, where $Q_i \in \mathcal{P}_i(\mathcal{X}) \cap \mathcal{P}_{i-1}^{\perp}(\mathcal{X})$, and so

$$\begin{aligned}
\|Q(\mathcal{X})\|_q &= \left\| T_\eta \left( \sum_{i=0}^{d} \eta^{-i} Q_i(\mathcal{X}) \right) \right\|_q \\
&\leq \left\| \sum_{i=0}^{d} \eta^{-i} Q_i(\mathcal{X}) \right\|_2 \\
&= \left( \sum_{i=0}^{d} \eta^{-2i} \|Q_i(\mathcal{X})\|_2^2 \right)^{\frac{1}{2}} && \text{(because } Q_i \perp Q_j \text{ for all } i \neq j\text{)} \\
&\leq \left( \sum_{i=0}^{d} \eta^{-2d} \|Q_i(\mathcal{X})\|_2^2 \right)^{\frac{1}{2}} && \text{(because } \eta < 1\text{)} \\
&= \eta^{-d} \left( \sum_{i=0}^{d} \|Q_i(\mathcal{X})\|_2^2 \right)^{\frac{1}{2}} \\
&= \eta^{-d} \|Q\|_2
\end{aligned}$$

□

**Exercise 5 (Gross '68)** *(2pts) Prove that the Orenstein-Uhlenbech operator $T_\eta$ is $(p, q, \eta)$ hypercontractive for all $(p, q, \eta)$ for which $\{-1, 1\}_0$ is $(p, q, \eta)$ hypercontractive. Recall that the Orenstein-Uhlenbech operator $T_\eta : L^2(\gamma_n) \to L^2(\gamma_n)$ is defined as follows:*

$$(T_\eta f)(x) = \mathbb{E}_{y \sim \gamma_n} \left[ f \left( \eta x + \sqrt{1 - \eta^2} y \right) \right]$$

## 2 Central Limit Theorem and Generalizations

**Exercise 6** *(1 pt) Let $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k)$ be a Gaussian vector and $f(\mathcal{N})$ a degree $d$ polynomial. Show that there exists a sequence $\{f_n\}$, where, for all $n$, $f_n$ is a multilinear polynomial on $\{-1, 1\}_0^{ndk}$, which converges in distribution to $f(\mathcal{N})$ as $n \to \infty$.*

**Theorem 7 (approach due to Linderberg)** *Let $q \in (2, 3]$, $\beta < \infty$, and let $X_1, \ldots, X_n$ be independent random variables satisfying $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = 1$, $\mathbb{E}[|X_i|^q] \leq \beta < \infty$. Also, let*

$$Q(X_1, \ldots, X_n) = \sum_{S \subseteq [n]} \left( c_S \prod_{i \in S} X_i \right)$$

*be a multi-linear polynomial of degree $d$ satisfying $\sum_{S\neq\emptyset} c_S^2 = 1$. Then, if $\Delta : \mathbb{R} \to [0,1]$ is non-decreasing with $\Delta(0) = 0$, $\Delta(1) = 1$, $A = \sup |\Delta^{(3)}(x)| < \infty$, $\Delta_r(x) = \Delta(\frac{x}{r})$, the following holds:*

$$|\mathbb{E}\left[\Delta_r(Q(X_1, X_2, \ldots, X_n))\right] - \mathbb{E}\left[\Delta_r(Q(G_1, \ldots, G_n))\right]| \leq A \cdot O_q\left(r^{-q}\beta^d \sum_i \left(\sum_{S:i\in S} c_S^2\right)^{q/2}\right)$$

*where $G_1, G_2, \ldots, G_n$ are independent Gaussian random variables and the constant hidden in the $O$ notation of the right hand side depends on $q$.*

The reason why the above theorem can be seen as a generalization of the central limit theorem is that, as function $\Delta(\cdot)$ approaches the step function $H(x) = \frac{1}{2}[1+sign(x-1)]$, the expectation $\mathbb{E}\left[\Delta_r(Q(X_1, X_2, \ldots, X_n))\right]$ approaches the probability $\mathbf{P}[Q(X_1, X_2, \ldots, X_n) \geq r]$ and, similarly, $\mathbb{E}\left[\Delta_r(Q(G_1, \ldots, G_n))\right]$ approaches $\mathbf{P}[Q(G_1, \ldots, G_n) \geq r]$. We will see the proof of the theorem in the next lecture. For now, we state and prove the following lemma.

**Lemma 8** *Let $q \in (2,3]$ and $Y$, $Z$ be random variables satisfying:*

- $\mathbb{E}[Y] = \mathbb{E}[Z]$
- $\mathbb{E}[Y^2] = \mathbb{E}[Z^2]$, $\mathbb{E}[|Y|^q], \mathbb{E}[|Z|^q] < \infty$

*Then for all $x$:*

$$|\mathbb{E}[\Delta_r(x+Y)] - \mathbb{E}[\Delta_r(x+Z)]| \leq Ar^{-q}\left(\mathbb{E}[|Y|^q] + \mathbb{E}[|Z|^q]\right)$$

*where $\Delta_r(\cdot)$ is the function defined in the statement of theorem 7.*

**Proof:** Since $\Delta''(0^-) = 0$, $\Delta''(1^+) = 0$ and $\Delta''(\cdot)$ is continuous it follows that $\Delta''(0) = 0$ and $\Delta''(1) = 0$. Hence

$$\sup_{0\leq x\leq 1/2} \left|\Delta''(x)\right| = \sup_{0\leq x\leq 1/2} \left|\int_0^x \Delta'''(t)dt\right| \leq \frac{A}{2}$$

$$\sup_{1/2\leq x\leq 1} \left|\Delta''(x)\right| = \sup_{1/2\leq x\leq 1} \left|\int_1^x \Delta'''(t)dt\right| \leq \frac{A}{2}$$

and so, trivially,

$$\sup_x \left|\Delta''(x)\right| \leq \frac{A}{2}.$$

Therefore, for all $x, y \in \mathbb{R}$:

$$
\begin{aligned}
\left|\Delta''(x) - \Delta''(y)\right| &= A\left|\frac{\Delta''(x)}{A} - \frac{\Delta''(y)}{A}\right| \\
&\leq A\left|\frac{\Delta''(x)}{A} - \frac{\Delta''(y)}{A}\right|^{q-2} \\
&= A^{3-q}\left|\Delta''(x) - \Delta''(y)\right|^{q-2} \\
&= A^{3-q}\left|\int_y^x \Delta'''(t)dt\right|^{q-2} \\
&\leq A^{3-q}\left(A|x-y|\right)^{q-2} \\
&\leq A|x-y|^{q-2}.
\end{aligned}
$$

This implies the following

$$
\left|\Delta_r''(x) - \Delta_r''(y)\right| = \frac{1}{r^2}\left|\Delta''\left(\frac{x}{r}\right) - \Delta''\left(\frac{y}{r}\right)\right| \leq \frac{A}{r^2}\left|\frac{x}{r} - \frac{y}{r}\right|^{q-2} \leq Ar^{-q}|x-y|^{q-2}.
$$

Now, if we denote by $\varphi(v) = \mathbb{E}[\Delta_r(x+vY)] - \mathbb{E}[\Delta_r(x+vZ)]$, $0 \leq v \leq 1$, we have:

- $\varphi(0) = 0$
- $\varphi'(v) = \mathbb{E}[Y\Delta_r'(x+vY)] - \mathbb{E}[Z\Delta_r'(x+vZ)]$
- $\varphi'(0) = \mathbb{E}[Y\Delta_r'(x)] - \mathbb{E}[Z\Delta_r'(x)] = 0$
- $\varphi''(v) = \mathbb{E}[Y^2\Delta_r''(x+vY)] - \mathbb{E}[Z^2\Delta_r''(x+vZ)]$

Therefore, we can write

$$
\begin{aligned}
|\varphi''(v)| &= \left|\mathbb{E}[Y^2(\Delta_r''(x+vY) - \Delta_r''(x))] - \mathbb{E}[Z^2(\Delta_r''(x+vZ) - \Delta_r''(x))]\right| \\
&\leq \mathbb{E}\left[\left|Y^2(\Delta_r''(x+vY) - \Delta_r''(x))\right|\right] + \mathbb{E}\left[\left|Z^2(\Delta_r''(x+vZ) - \Delta_r''(x))\right|\right] \\
&\leq Ar^{-q}\left(\mathbb{E}\left[Y^2|vY|^{q-2}\right] + \mathbb{E}\left[Z^2|vZ|^{q-2}\right]\right) = \\
&\leq Ar^{-q}v^{q-2}\left(\mathbb{E}\left[|Y|^q\right] + \mathbb{E}\left[|Z|^q\right]\right)
\end{aligned}
$$

After integrating twice with respect to $v$ and plugging in $v = 1$, we get

$$
|\varphi(1)| \leq Ar^{-q}\left(\mathbb{E}\left[|Y|^q\right] + \mathbb{E}\left[|Z|^q\right]\right).
$$

□