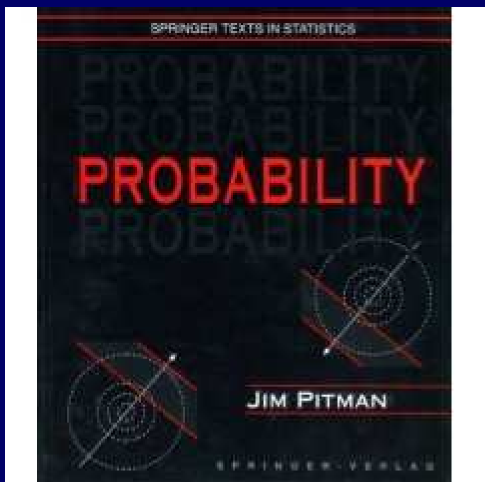
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

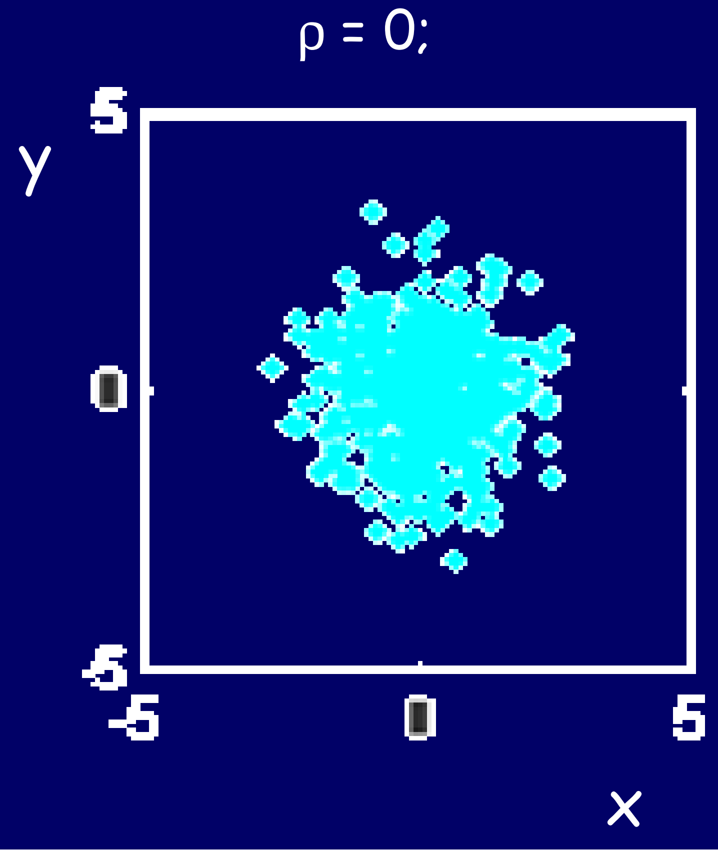
Probability

Sections 6.5

Bivariate Normal

Let (X,Y) be independent Normal variables.

The joint density: $f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2+y^2)}$



Bivariate Normal

Question: What is the joint distribution of (X,Y)

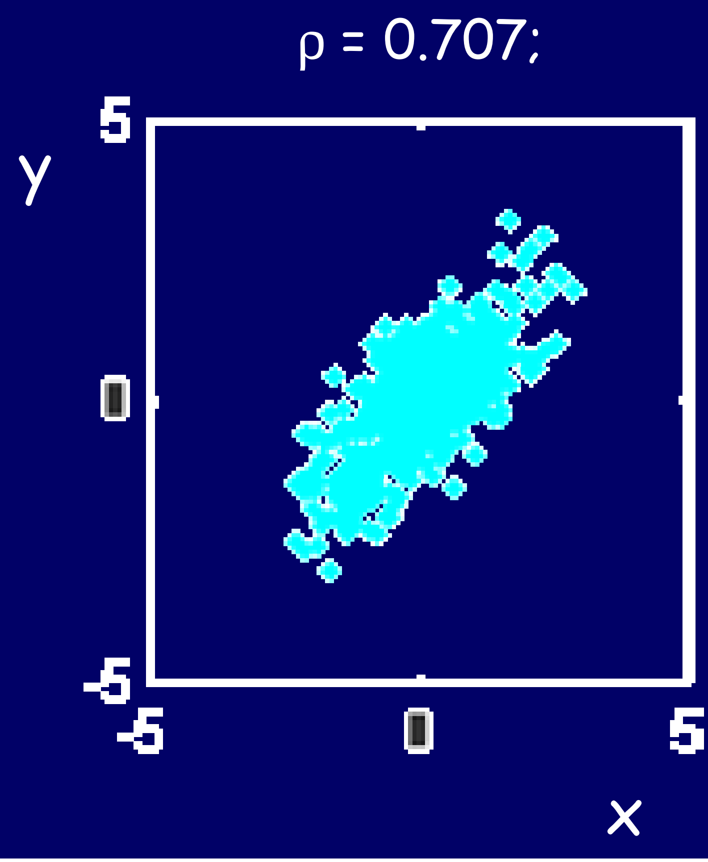
Where X = child height and Y = parent height?

- We expect X and Y to be Gaussian
- However, X and Y are not independent:

$\rho(X,Y) > 0$.

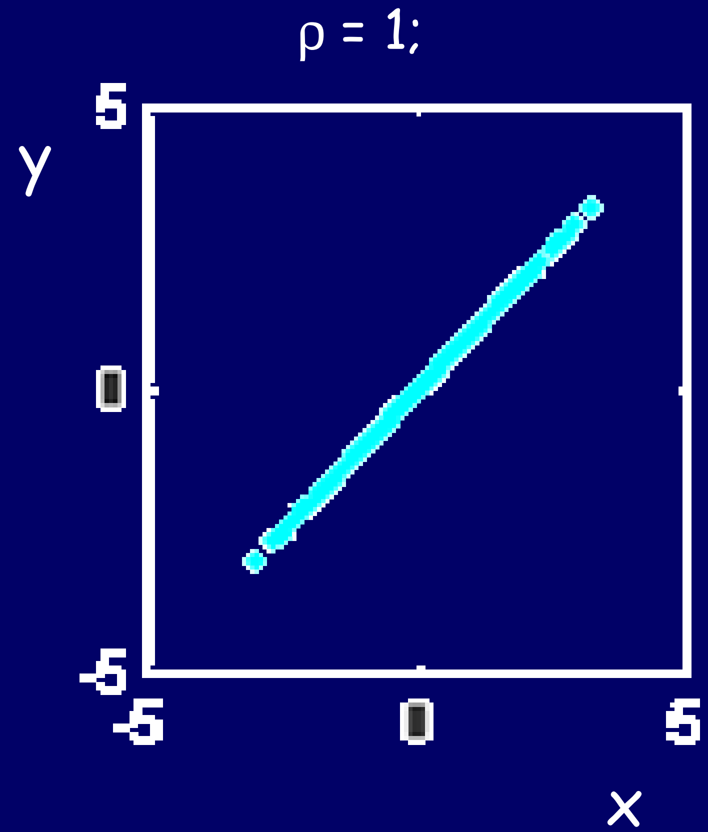
Bivariate Normal

Intuitive sample of Gaussian (X,Y) with ρ = 0.7



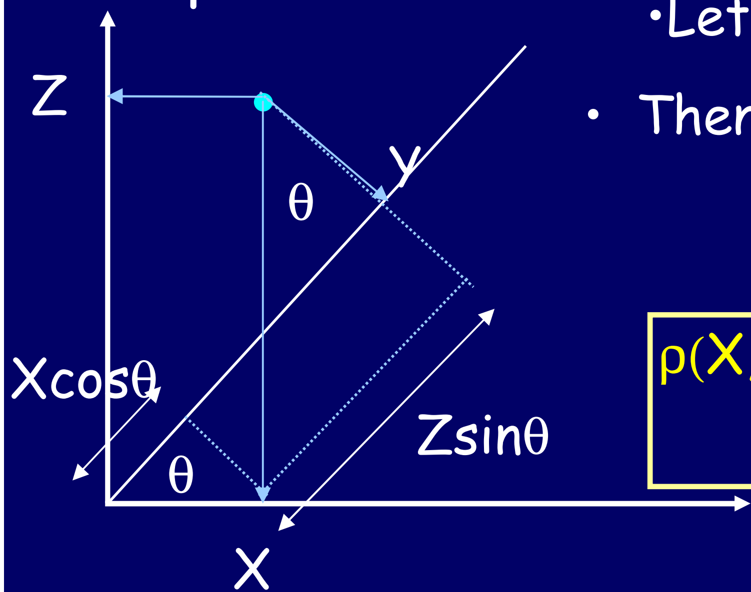
Bivariate Normal

Intuitive sample of X,Y with $\rho(X,Y) = 1$.



Construction of Bivariate Normal

Construction of correlated Gaussians: Let $X,Z \sim N(0,1)$ be independent.



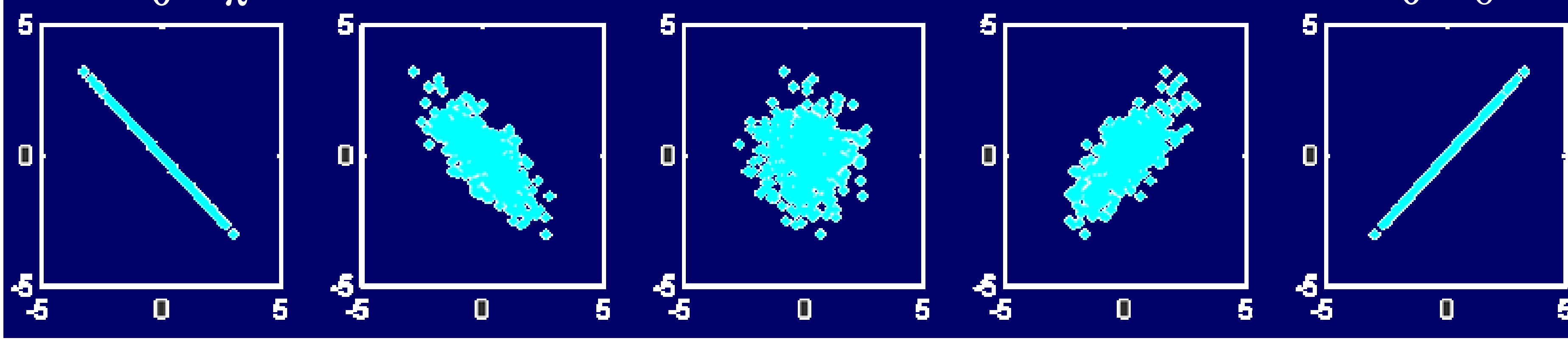
- Let: $Y = X\cos\theta + Z\sin\theta$;
- Then: $E(Y) = E(X) = E(Z) = 0$

  $Var(Y) = \cos^2\theta + \sin^2\theta = 1$

  $SD(Y) = SD(X) = SD(Z) = 1$ $\quad Y \sim N(0,1)$

$$\boxed{\begin{aligned}\rho(X,Y) &= E(XY) = E(X^2)\cos\theta + E(XZ)\sin\theta \\ &= \cos\theta\end{aligned}}$$

$\rho = -1;$ $\theta = \pi$ | $\rho = -0.707;$ $\theta = 3\pi/4$ | $\rho = 0;$ $\theta = \pi/2$ | $\rho = 0.707;$ $\theta = \pi/4$ | $\rho = 1;$ $\theta = 0$



Standard Bivariate Normal

Def: We say that X and Y have standard bivariate normal distribution with correlation ρ if and only if

$$Y = \rho X + \sqrt{1-\rho^2}\,Z$$

Where X and Z are independent N(0,1).

Claim: If (X,Y) is ρ correlated bivariate normal then:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\}$$

Marginals: $X \sim N(0,1)$; $Y \sim N(0,1)$

Conditionals: $X|Y=y \sim N(\rho\, y, 1-\rho^2)$; $Y|X=x \sim N(\rho\, x, 1-\rho^2)$

Independence: X,Y are independent if and only if $\rho = 0$

Symmetry: (Y,X) is ρ correlated bivariate normal.

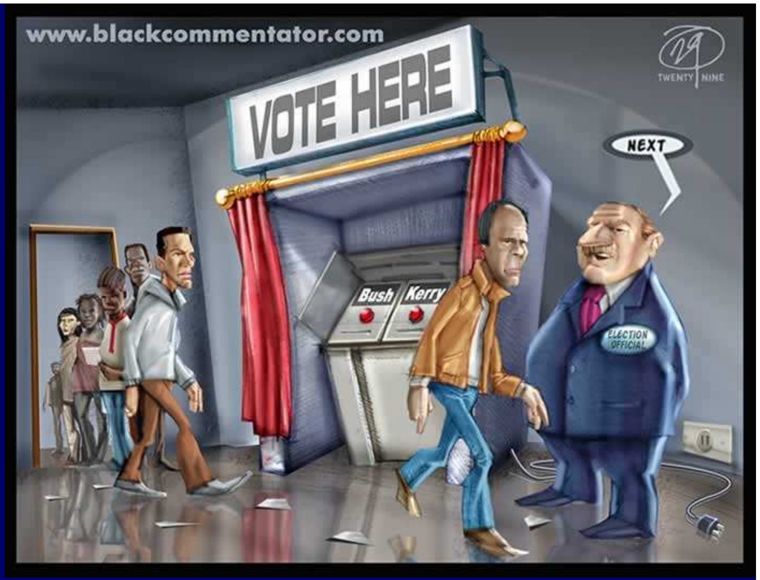
Bivariate Normal Distribution

<u>Definition:</u> We say that the random variables U and V have bivariate normal distribution with parameters μ_U, μ_V, $\sigma^2{}_U$, $\sigma^2{}_V$ and ρ if and only if the standardized variables

$$X=(U - \mu_U)/\sigma_U \qquad Y=(V - \mu_V)/\sigma_V$$

have standard bivariate normal distribution with correlation ρ.

Errors in Voting Machines



There are two candidates in a certain election: Mr.B and Mr.K. We use a voting machine to count the votes. Suppose that the voting machine at a particular polling station is somewhat capricious – it flips the votes with probability ε.

A voting Problem

•Consider a vote between two candidates where:

•At the morning of the vote:
Each voter tosses a coin and votes according to the outcome.

•Assume that the winner of the vote is the candidate with the majority of votes.

• In other words let $V_i \in \{\pm 1\}$ be the vote of voter i. So

- if $V = \sum_i V_i > 0$ then $+1$ wins;
- otherwise -1 wins.

A mathematical model of voting machines

Which voting schemes are more robust against noise?

Simplest model of noise: The voting machine flips each vote independently with probability ε.

Intended vote

Registered vote

-1

prob ε → +1

prob $1-\varepsilon$ → -1

1

prob ε → -1

prob $1-\varepsilon$ → +1

On Errors in voting machines

Question: Let V_i = intended vote i.

W_i = registered vote i.

What is dist of $V = \sum_{i=1}^{n} V_i$, $W = \sum_{i=1}^{n} W_i$ for large n?

Answer: $V \sim N(0,n)$; $W \sim N(0,n)$.

Question: What is $P[V > 0]$? $P[W > 0]$?

Answer: $\sim\frac{1}{2}$.

Question: What is the probability that machine errors flipped the elections outcome?

Answer: $P[V > 0, W < 0] + P[V < 0, W > 0]$?

$= 1 - 2\, P[V > 0, W > 0]$.

On Errors in voting machines

Answer continued: Take $(X,Y) = (V,W)/n^{1/2}$.

Then (X,Y) is bivarite-normal where $SD(X) = SD(Y) = 1$ and

$\rho = \rho(X,Y) = \rho(V,W)/n = \rho(V_i,W_i) = 1 - 2\,\varepsilon$.

Need to find $1\text{-}2P[X > 0,\ Y > 0]$.

Answer continued: Let $\rho = \cos\theta$. Then we need to find:

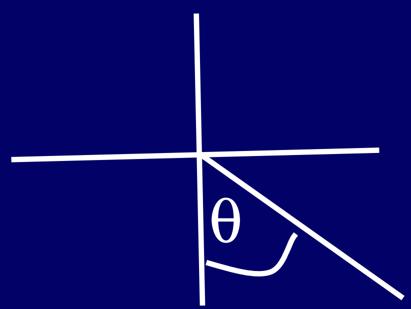
$$P[X > 0,\ Y > 0] = P[X \cos 0 + Z \sin 0 > 0,\ X \cos\theta + Z \sin\theta > 0] =$$

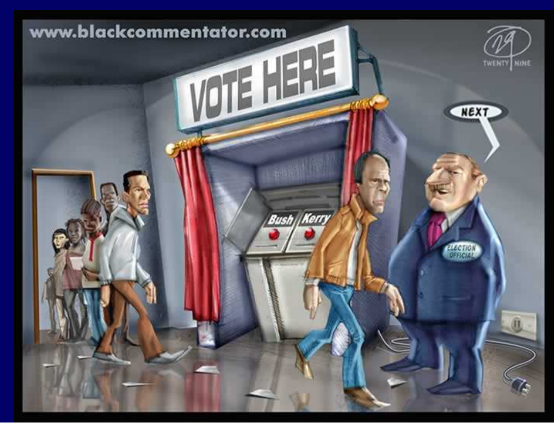
$$= P[X > 0,\ Z > -(\operatorname{ctg}\theta)\, X] = (\pi - \theta)/2\pi =$$

$$= \tfrac{1}{2}(1-(\arccos\rho)/\pi).$$

So $1\text{-}2P[X > 0,\ Y > 0] = (\arccos\rho)/\pi$

$$= (\arccos(1-2\varepsilon))/\pi$$



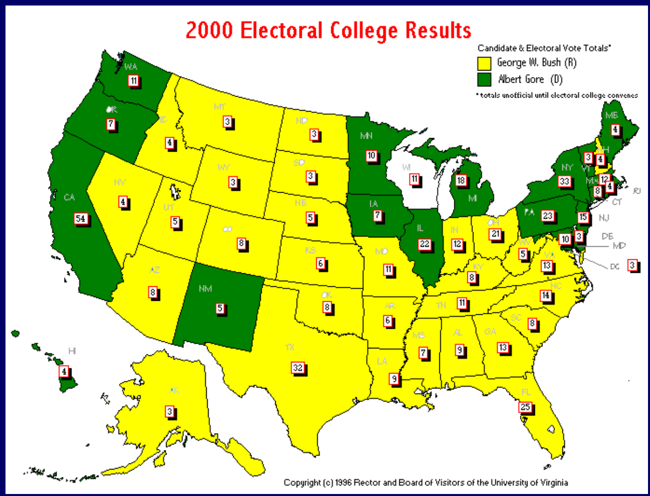


Majority and Electoral College

- Probability of error ~ $\varepsilon^{1/2}$

- Result is essentially due to Sheppard (1899): "On the application of the theory of error to cases of normal distribution and normal correlation".

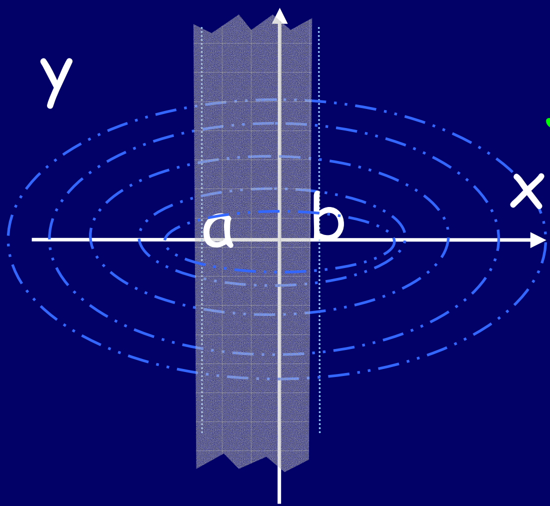
- For $n^{1/2} \times n^{1/2}$ electoral college f $\varepsilon^{1/4}$



Conditional Expectation given an interval

Suppose that (X,Y) has the Standard Bivariate Normal Distribution with correlation ρ.

Question: For a < b, what is the E(Y|a < X <b)?



Solution:

$$E(Y|a < X < b) = \frac{\int_a^b \int_{-\infty}^{\infty} y f(x,y) dy dx}{\int_a^b \int_{-\infty}^{\infty} f(x,y) dy dx}$$

$$= \frac{\int_a^b \int_{-\infty}^{\infty} y f_Y(y|X=x) f_X(x) dy dx}{\int_a^b f_X(x) dx}$$

$$= \frac{\int_a^b E(Y|X=x) f_X(x) dy dx}{\int_a^b f_X(x) dx}.$$

Conditional Expectation given an interval

Solution: $E(Y \mid a < X < b) = \int_a^b E(Y \mid X = x)\, f_X(x) \,/\, \int_a^b f_X(x)dx$

We know that $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\int_a^b f_X(x)\, dx = \Phi(b) - \Phi(a)$.

Since $Y = \rho X + \sqrt{1-\rho^2} Z$, where X & Z are independent

We have: $E(Y|X{=}x) = E(\rho X + \sqrt{1-\rho^2}Z | X{=}x) = \rho x$

$$\boxed{\begin{aligned} E(Y|a<X<b) &= \frac{\rho}{\sqrt{2\pi}} \int_a^b x\, e^{-\frac{x^2}{2}} dx \,/\, \left(\Phi(b)-\Phi(a)\right) \\ &= \frac{\rho}{\sqrt{2\pi}} \left[e^{-\frac{a^2}{2}} - e^{-\frac{b^2}{2}}\right] / \left(\Phi(b)-\Phi(a)\right) \end{aligned}}$$

Linear Combinations of indep. Normals

Claim: Let $V = \sum_{i=1}^{n} a_i Z_i$, $W = \sum_{i=1}^{n} b_i Z_i$ where Z_i are independent normal variables $N(\mu_i,\sigma_i^2)$. Then (V,U) is bivariate normal.

Problem: calculate $\mu_V,\mu_W,\sigma_V,\sigma_W$ and ρ