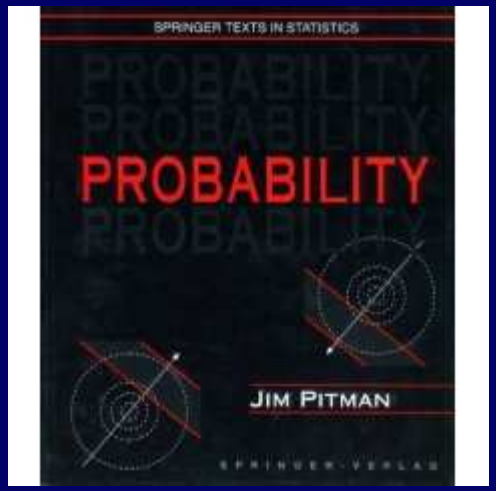
Introduction to probability

Stat 134 FAll 2005

Berkeley



Lectures prepared by:
Elchanan Mossel
Yelena Shvets

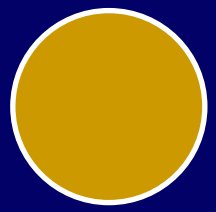
Follows Jim Pitman's book:

Probability

Sections 6.3

Conditional density

Example: (X,Y) uniform in the unit disk centered at 0.



Question: $P(X > 0.1 \mid Y > 0.1) = ?$

Answer: $$P(X > 0.1 \mid Y > 0.1) = \frac{P(X > 0.1 \;\&\; Y > 0.1)}{P(Y > 0.1)}$$

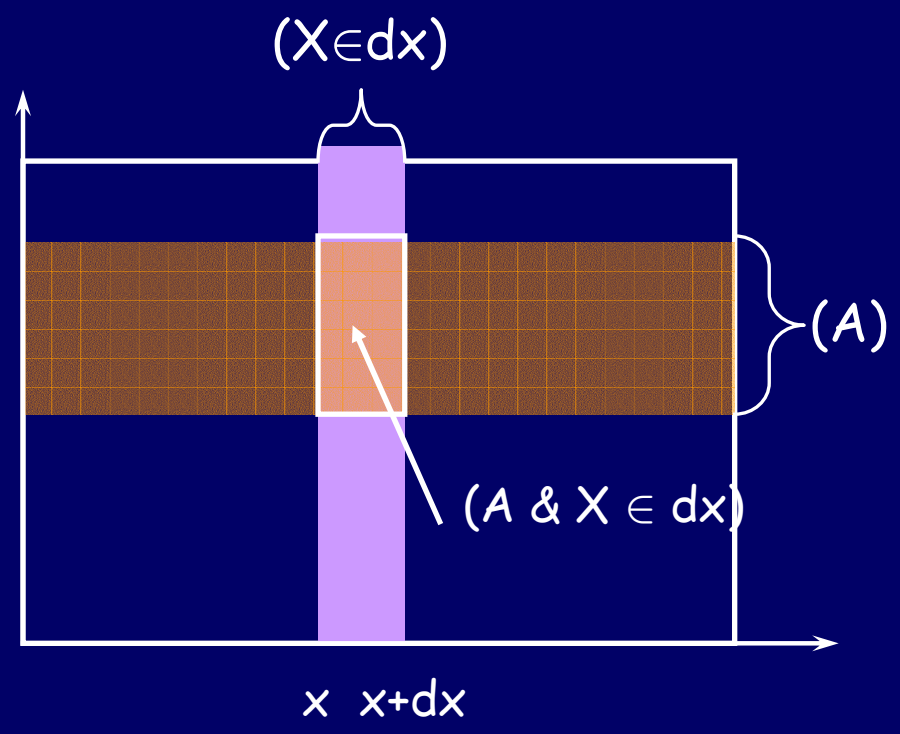
Question: $P(X > 0.1 \mid Y = 0.1) = ?$

Try: $$P(X > 0.1 \mid Y = 0.1) = \frac{P(X > 0.1 \;\&\; Y = 0.1)}{P(Y = 0.1)} = \frac{0}{0}!$$

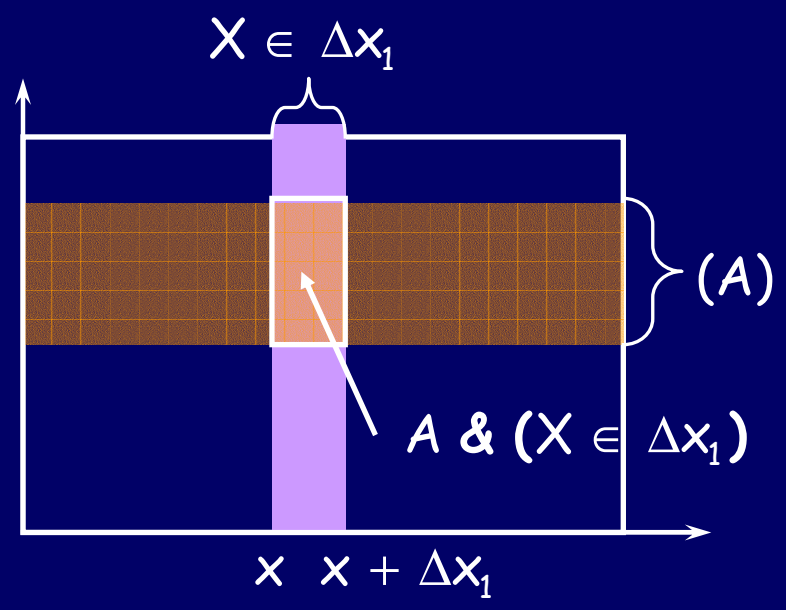
Infinitesimal Conditioning Formula

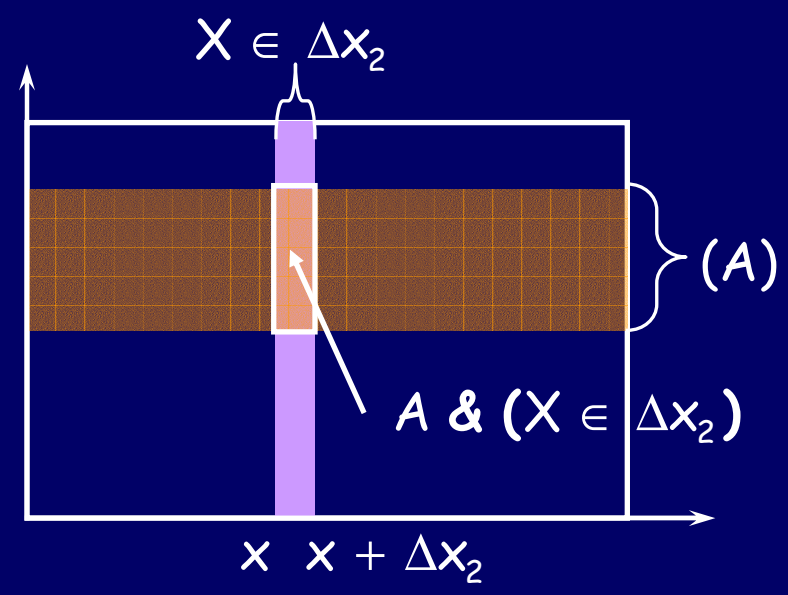
Solution: Replace by infinitesimal ratio:

$$P[A|X=x] = \lim_{dx \to 0} \frac{P[A, X \in (x, x+dx)]}{P[X \in (x, x+dx)]}$$



Infinitesimal Conditioning Formula





$$P(A \mid X = x) = \lim_{\Delta x \to 0} P(A \mid X \in \Delta x)$$

$$= \lim_{\Delta x \to 0} \frac{P(A \;\&\; X \in \Delta x)}{P(X \in \Delta x)}$$

Conditional Density

Claim: Suppose (X,Y) have a joint density $f(x,y)$, and x is such that $f_X(x) > 0$, the conditional density of Y given $X=x$ is a probability density defined by

$$f_Y(y \mid X = x) = \frac{f(x, y)}{f_X(x)}$$

"Proof":

$P(a \leq Y \leq b \mid X = x) =$

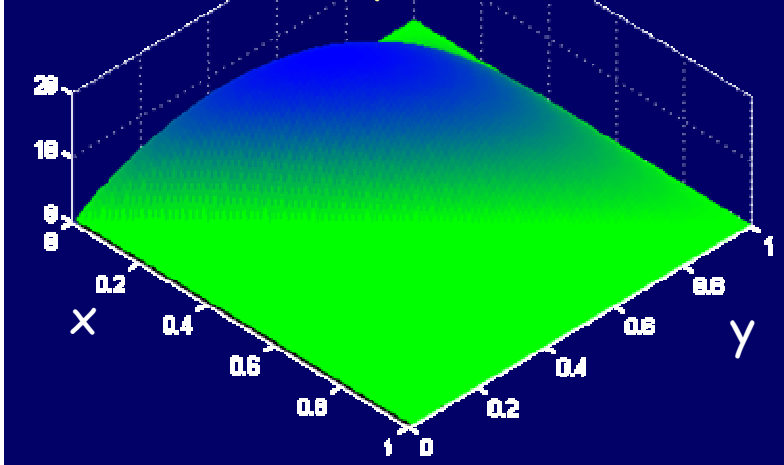
$\lim P(a \leq Y \leq b \mid X \in (x, x+dx)) =$

$\lim P(a \leq Y \leq b, X \in (x, x+ dx))/ f_X(x)\, dx =$

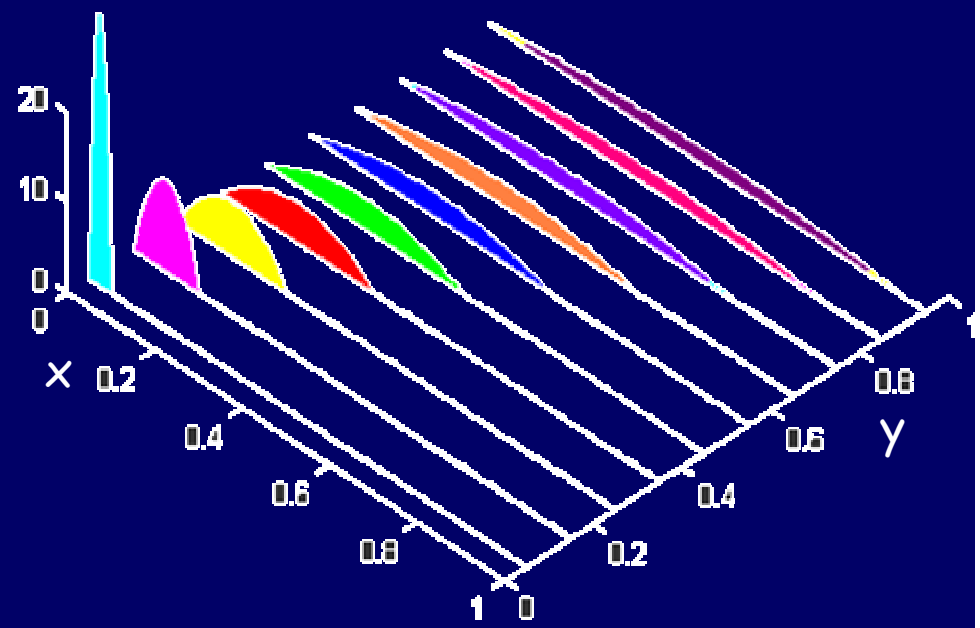
$\lim \int_a^b f(x,y)\, dy\, dx \,/\, f_X(x)\, dx = \int_a^b f_Y(y \mid X = x)\, dy$

Conditional Density

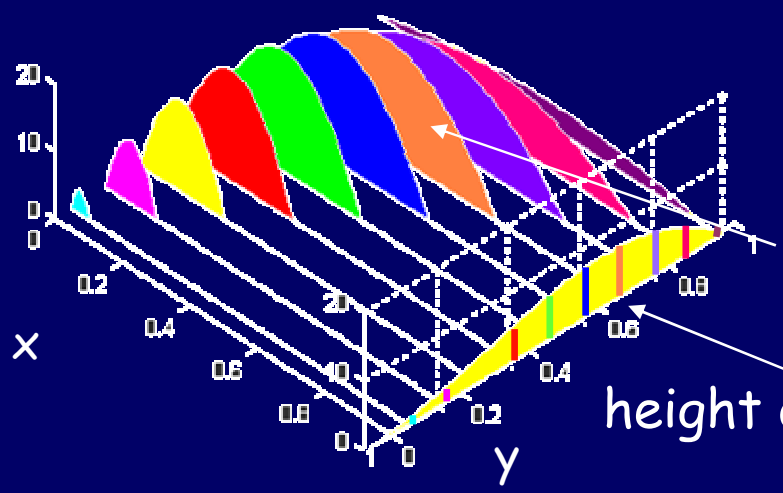
Joint density of (X,Y):
f(x,y):



Renormalized slice for given y =
conditional density P[X | Y=y],
The area is of each slice is 1.



Slices through density for fixed y's.



Area of slice at y
=
height of marginal distribution at y.

Conditioning Formulae

Discrete	Continuous

- Multiplication Rule:

$P(x,y) = P(Y=y\|X=x)P(X=x)$	$f(x,y) = f_Y(y\|X=x)f_X(x)$

- Division Rule:

$P(Y=y\|X=x) = \dfrac{P(X=x,Y=y)}{P(X=x)}$	$f_Y(y\|X=x) = \dfrac{f(x,y)}{f_X(x)}$

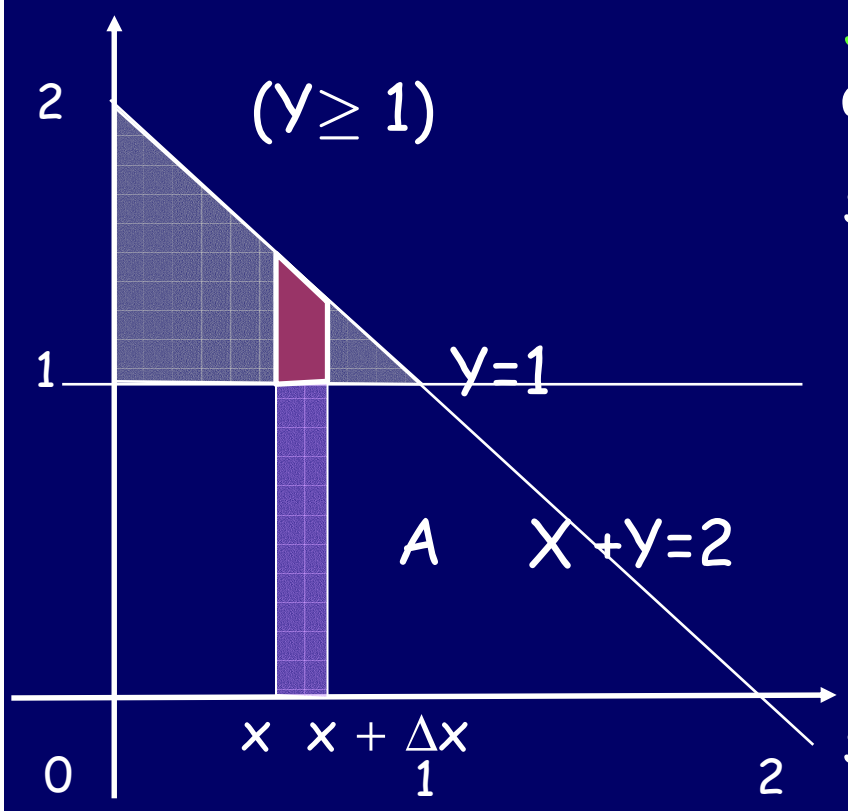
- Bayes' Rule:

$P(X=x\|Y=y) = \dfrac{P(Y=y\|X=x)P(X=x)}{P(Y=y)}$	$f_X(x\|Y=y) = \dfrac{f_Y(y\|X=x)f_X(x)}{f_Y(y)}$

Example: Uniform on a triangle.

Suppose that a point (X,Y) is chosen uniformly at random from the triangle A: $\{(x,y): 0 \le x, 0 \le y, x+y \le 2\}$.

Question: Find $P(Y \ge 1 \mid X = x)$.



Solution 1: Since $Area(A) = 2$, for all S, $P(S) = Area(S)/2$.

So for $0 \le x < 1$:

$$P(X \in \Delta x) = \tfrac{1}{2}\Delta x\,(2-x-\tfrac{1}{2}\Delta x);$$

$$P(Y>1, X \in \Delta x) = \tfrac{1}{2}\Delta x\,(1-x-\tfrac{1}{2}\Delta x);$$

$$P(Y \ge 1 \mid X \in \Delta x) = P(Y>1, X \in \Delta x) \,/\, P(X \in \Delta x)$$

$$= (1-x-\tfrac{1}{2}\Delta x)/\,(2-x-\tfrac{1}{2}\Delta x);$$

$$\to (1-x)/(2-x) \text{ as } \Delta x \to 0.$$

So $P(Y \ge 1 \mid X = x) = (1-x)/(2-x)$

And for $x \ge 1$: $P(Y \ge 1 \mid X = x) = 0$.

Example: Uniform on a triangle.

Suppose that a point (X,Y) is chosen uniformly at random from the triangle A: $\{(x,y): 0 \le x, 0 \le y, x+y \le 2\}$.

Question: Find $P(Y \ge 1 \mid X = x)$.

Solution 2. using the conditional density $f_Y(y|X=x)$.

For the uniform distribution on a triangle of area 2,

$$f(x,y) = \begin{cases} 1/2, & \text{if } 0 \le x, 0 \le y, x+y \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

So for $0 \le x \le 2$, $$f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^{2-x} \frac{1}{2}\,dy = \frac{2-x}{2},$$

$$f_Y(y|X=x) = \begin{cases} \frac{f(x,y)}{f_X(x)} = \frac{1}{2-x} & \text{if } 0 \le y \le 2-x; \\ 0, & \text{otherwise.} \end{cases}$$

Given, X=x, the distribution of Y is uniform on (0,2-x).

$$P[Y \ge 1 | X = x] = \int_1^{2-x} \frac{1}{2-x} dx = \frac{1-x}{2-x}.$$

Example: Gamma and Uniform

Suppose that X has Gamma(2, λ) distribution, and that given X=x, Y has Uniform(0,x) distribution.

Question: Find the joint density of X and Y

Solution:

By definition of Gamma(2, λ) distr., $f_X(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{if } 0 < x; \\ 0, & \text{otherwise.} \end{cases}$

By definition of Unif(0,x), $f_Y(y|X=x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < y < x; \\ 0, & \text{otherwise.} \end{cases}$

By multiplication rule, $f(x,y) = f_Y(y|X=x) f_X(x) = \begin{cases} \lambda^2 e^{-\lambda x}, & \text{if } 0 < y < x; \\ 0, & \text{otherwise.} \end{cases}$

Question: Find the marginal density of Y.

Solution: $f_Y(y) = \int_0^\infty f(x,y)\,dx = \int_y^\infty \lambda^2 e{-}\lambda x\,dx = \lambda e^{-\lambda x}.$

So Y ~ Exp(λ).

Condition Distribution & Expectations

Discrete	Continuous

- Conditional distribution of Y given X=x;

$P(Y \in B \mid X = x) = \sum_{y \in B} P(Y = y \mid X = x)$	$P(Y \in B \mid X = x) = \int_B f_Y(y \mid X = x)\, dy$

- Expectation of a function g(X):

$E(g(Y) \mid X = x) = \sum_{all\, y} g(y) P(Y = y \mid X = x)$	$E(g(Y) \mid X = x) = \int g(y) f_Y(y \mid X = x)\, dy$

Independence

Conditional distribution of Y given X=x does not depend on x:

$$P(Y \in B | X=x) = P(Y \in B).$$

Conditional distribution of X given Y=y does not depend on y:

$$P(X \in A | Y=y) = P(X \in A).$$

The multiplication rule reduces to:

$$f(x,y) = f_X(x) f_Y(y).$$

Expectation of the product is the product of Expectations:

$$E(XY) = \int \int xy\, f_X(x)\, f_Y(y)\, dx\, dy = E(X)\, E(Y).$$

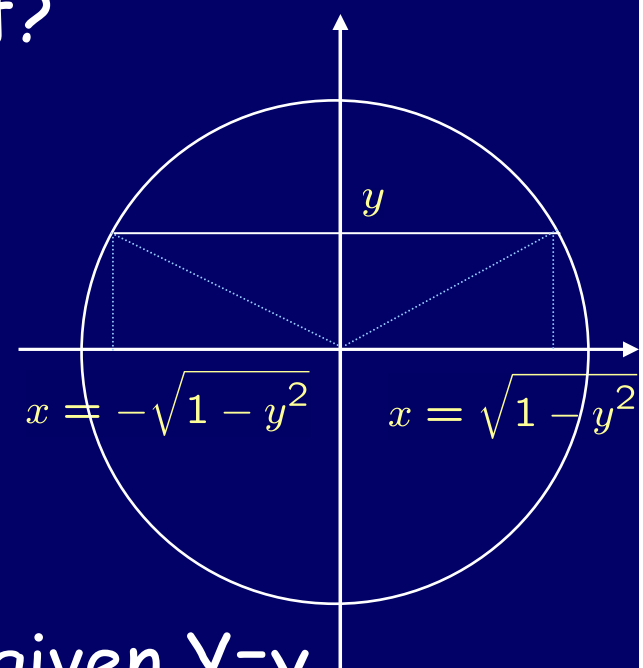
Example: (X,Y) Uniform on a Unit Circle.

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if } x^2 + y^2 < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Question: Are X and Y independent?

Solution:

$$f_X(x|Y=y) = \begin{cases} \frac{1}{2\sqrt{1-y^2}}, & \text{if } -\sqrt{1-y^2} < x < \sqrt{1-y^2}; \\ 0, & \text{otherwise.} \end{cases}$$



So the conditional distribution of X given Y=y depends on y and hence X and Y cannot be independent. However, note that for all x and y

E(X|Y = y) = E(X) = 0 and E(Y|X=x) = E(Y) = 0

Average Conditional Probabilities & Expectations

Discrete	Continuous

- Average conditional probability:

$P(B) = \sum_{all\, x} P(B\|X=x)P(X=x)$	$P(B) = \int P(B\|X=x) f_X(x)\, dx$
$P(Y=y) = \sum_{all\, x} P(Y=y\|X=x)P(X=x)$	$f_Y(y) = \int f_Y(y\|X=x) f_X(x)\, dx$

- Average conditional expectation:

$E(Y) = \sum_{all\, x} E(Y\|X=x)P(X=x)$	$E(Y) = \int E(Y\|X=x) f_X(x)\, dx$