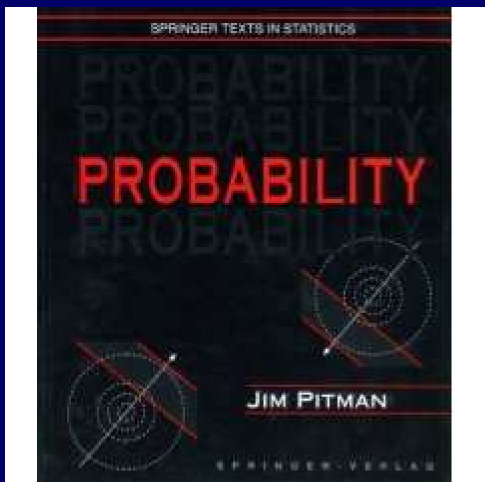
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Sections 6.1-6.2

of Heads in a Random # of Tosses

- Suppose a fair die is rolled and let N be the number on top.

N=5

- Next a fair coin is tossed N times and H, the number of heads is recorded:

H=3

Question:

Find the distribution of H, the number of heads?

of Heads in a Random Number of Tosses

Solution:

- The conditional probability for the H=h, given N=n is

$$P(H=h|N=n) = \binom{n}{h}\left(\frac{1}{2}\right)^n, \quad \binom{n}{h} = 0 \text{ if } n < h.$$

- By the rule of the average conditional probability and using P(N=n) = 1/6

$$P(H=h) = \sum_{n=1}^{6} P(H=h|N=n)P(N=n) = \frac{1}{6}\sum_{n=1}^{6}\binom{n}{h}\left(\frac{1}{2}\right)^n.$$

h	0	1	2	3	4	5	6
P(H=h)	63/384	120/384	99/384	64/384	29/384	8/384	1/384

Conditional Distribution of Y given X =x

Def: For each value x of X, the set of probabilities

$$P(Y=y|X=x)$$

where y varies over all possible values of Y, form a probability distribution, depending on x, called the conditional distribution of Y given X=x.

Remarks:

- In the example $P(H = h \mid N = n) \sim \text{binomial}(n, \frac{1}{2})$.
- The unconditional distribution of Y is the average of the conditional distributions, weighted by $P(X=x)$.

Conditional Distribution of X given Y=y

Remark: Once we have the conditional distribution of Y given X=x, using Bayes' rule we may obtain the conditional distribution of X given Y=y.

- Example: We have computed distribution of H given N = n:

$$P(H = h|N = n) = \binom{n}{h}\left(\frac{1}{2}\right)^n.$$

- Using the product rule we can get the joint distr. of H and N:

$$P(H = h \,\&\, N = n) = P(H = h|N = n)P(N = n) = \frac{1}{6}\binom{n}{h}\left(\frac{1}{2}\right)^n,$$

- Finally:

$$P(N = n|H = h) = \frac{P(H = h \,\&\, N = n)}{P(H = h)};$$

$$= \frac{\frac{1}{6}\binom{n}{h}\left(\frac{1}{2}\right)^n}{\frac{1}{6}\sum_{i=1}^{6}\binom{i}{h}\left(\frac{1}{2}\right)^i} = \frac{\binom{n}{h}}{\sum_{i=1}^{6}\binom{i}{h}2^{n-i}}.$$

Number of 1's before the first 6.

Example: Roll a die until 6 appears.

Let Y = # 1's and X = # of Rolls.

Question: What is the Distribution of Y?

Solution:

1 – Find distribution of X:

$$P(X = x) = \left(\frac{5}{6}\right)^{x-1} \frac{1}{6},$$

so X~Geom(1/6).

2- Find conditional distribution of Y given X=x.

$$P(Y = y|X = x) = \binom{x-1}{y} \left(\frac{1}{5}\right)^{y} \left(\frac{4}{5}\right)^{x-1-y},$$

so P(Y|X=x)~Bin(x-1,1/5).

Number of 1's before the first 6.

3- Use the rule of average conditional probabilities to compute the unconditional distribution of Y.

$$P(Y=y) = \sum_{x=y+1}^{\infty} P(Y=y|X=x)P(X=x)$$

$$= \sum_{x=y+1}^{\infty} \binom{x-1}{y} \left(\frac{1}{5}\right)^{y} \left(\frac{4}{5}\right)^{x-1-y} \left(\frac{5}{6}\right)^{x-1} \frac{1}{6};$$

$$= \frac{1}{6}\left(\frac{1}{4}\right)^{y} \sum_{x=y+1}^{\infty} \binom{x-1}{y} \left(\frac{4}{6}\right)^{x-1};$$

$$= \frac{1}{6}\left(\frac{1}{4}\right)^{y} \frac{1}{y!}\left(\frac{4}{6}\right)^{y} \sum_{x=y+1}^{\infty} (x-1)(x-2)\ldots(x-y) \left(\frac{4}{6}\right)^{x-y-1};$$

$$= \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} \sum_{x=y}^{\infty} (x)(x-1)\ldots(x-y+1) \left(\frac{4}{6}\right)^{x-y};$$

$$= \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} \left[\frac{d^y}{dr^y} \sum_{x=0}^{\infty} (r)^x\right]_{r=\frac{4}{6}} = \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} \left[\frac{d^y}{dr^y} \frac{1}{1-r}\right]_{r=\frac{4}{6}};$$

$$= \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} y! \left(1-\frac{4}{6}\right)^{-(y+1)} = \left(\frac{1}{2}\right)^{y+1}.$$

So Y = G-1, where
G = Geom(1/2)

Conditional Expectation Given an Event

Definition: The conditional expectation of Y given an event A, denoted by E(Y|A), is the expectation of Y under the conditional distribution given A:

$$E(Y|A) = \sum_{\text{all } y} y \, P(Y=y|A).$$

Example: Roll a die twice; Find E(Sum| No 6's).

X+Y	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

E(X+Y|No 6's) =

(2*1+ 3*2+ 4*3 + 5*4 +6*5 + 7*4 + 8*3 + 9*2 + 10*1)/25

= 6

Linearity of Conditional Expectation

Claim:

For any set A:

$$E(X + Y \mid A) = E(X|A) + E(Y|A).$$

Proof: $E(X + Y|A) = \sum_{all\ (x,y)} (x+y)\, P(X=x\ \&\ Y=y|A)$

$= \sum_{all\ x} x \sum_{all\ y} P(X=x\ \&\ Y = y|\ A)$

$+ \sum_{all\ y} y \sum_{all\ x} P(Y=y\ \&\ X = x \mid A)$

$= \sum_{all\ x} x\, P(X=x \mid A)$

$+ \sum_{all\ y} y\, P(Y=y \mid A)$

$= E(X|A) + E(Y|A).$

Using Linearity for 2 Rolls of Dice

Example: Roll a die twice; Find E(Sum| No 6's).

$E(X+Y|X\neq 6 \ \& \ Y\neq 6) = E(X|X\neq 6 \ \& \ Y\neq 6)+E(Y|X\neq 6 \ \& \ Y\neq 6)$

$=2\ E(X|X\neq 6)$

$= 2(1 + 2 + 3 + 4 + 5)/5$

$=6$

Conditional Expectation

Claim: For any Y with $E(Y) < \infty$ and any discrete X,

$$E(Y) = \sum_{\text{all } x} E(Y|X=x)\, P(X=x).$$

This is called the "rule of average conditional expectation".

Definition of **conditional expectation**:

The conditional expectation of Y given X, denoted by $E(Y|X)$, is a random variable that depends on X. It's value, when $X = x$, is $E(Y|X=x)$.

Compact form of rule of average conditional E's :

$$E[Y] = E[E(Y|X)]$$

Conditional Expectations: Examples

Example:

Let N = number on a die, H = # of heads in N tosses.

Question: Find E(H|N).

Solution:

We have seen that $P(H|N=n) \sim Bin(n, \frac{1}{2})$. So $E(H|N=n) = n/2$.
and therefore $E(H|N) = N/2$.

Question: Find E(H).

Solution: This we can do by conditioning on N:

$E(H) = E(E(H|N)) = E(N/2) = 3.5/2 = 1.75.$

Conditional Expectation - Examples

Roll a die until 6 comes up.

N = # of rolls; Y = # of 1's.

Question: Compute $E[Y]$.

Solution: We will condition on N: $E[Y] = E[\, E[Y|N]]$.

Then: $E[Y|N=n] = (n-1)/5$.

$N \sim Geom(1/6)$ and $E[N] = 6$. Thus

$$E[Y] = E[E[Y|N]] = E[(N-1)/5] = 1.$$

Note: We've seen before then $Y = G - 1$ where $G \sim Geom(1/2)$. This also gives: $E[Y] = 2-1 = 1$.

Conditional Expectation

Roll a die until 6 comes up.

S=sum of the rolls; N = # of rolls.

Question: Compute $E[S]$,

Solution: We will condition on N: $E[S] = E[\, E[S|N]]$.

Need: $E[S|N=n]$;

Let X_i denote the # on the i^{th} roll die.

$E\,[S|N=n] = E(X_1 + \ldots + X_n) = 3(n-1) + 6$;

$E[S|N] = 3N+3$

$N \sim Geom(1/6)$ and $E[N] = 6$. Thus

$$E[S] = 3E[N]+3 = 21$$

Success counts in overlapping series of trials

Example: Let S_n number of successes in n independent trials with probability of success p.

Question: Calculate $E(S_m|S_n=k)$ for $m \leq n$

Solution: $S_m = X_1 + \ldots + X_m$, where $X_j = 1$(success on j^{th} trial).

Since $S_m = X_1 + \ldots + X_m$ we have

$$E[S_m|S_n = k] = \sum_{j=1}^{m} E[X_j|S_n = k] \text{ and}$$

$$E[X_j|S_n = k] = P[X_j = 1|S_n = k] =$$

$$\frac{P[X_j = 1, S_n = k]}{P[S_n = k]} = \frac{p\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{k}{n}$$

So $E[S_m] = k\, m / n$

Success counts in overlapping series of trials

Example: Let S_n number of successes in n independent trials with probability of success p.

Question: Calculate $E(S_m|S_n=k)$ for $m \leq n$

Solution: $S_m = X_1 + \dots + X_m$, where $X_j = 1(\text{success on } j^{th} \text{ trial})$.

Alternatively, we note that $E[X_i \mid S_n = k] = E[X_j \mid S_n = k]$ for all i and j by symmetry.

However $E[\sum_{i=1}^n X_i \mid S_n = k] = E[S_n \mid S_n = k] = k$.

Therefore $E[X_i \mid S_n = k] = k/n$ and $E[S_m \mid S_n = k] = km/n$

Conditional Expectation

Comment: Conditioned variables can be treated as constants when taking expectations:

$$E[g(X,Y)|X=x] = E[g(x,Y)|X=x]$$

Example:

$$E[XY|X=x] = E[xY|X=x] = xE[Y|X=x];$$

so as random variables:

$$E[XY|X] = XE[Y|X].$$

Example: $E[aX + bY|X] = aX + bE[Y|X];$

Question: Suppose X and Y are independent, find $E[X+Y|X]$.

Solution: $E(X+Y|X) = X + E(Y|X) = X + E(Y)$, by independence.