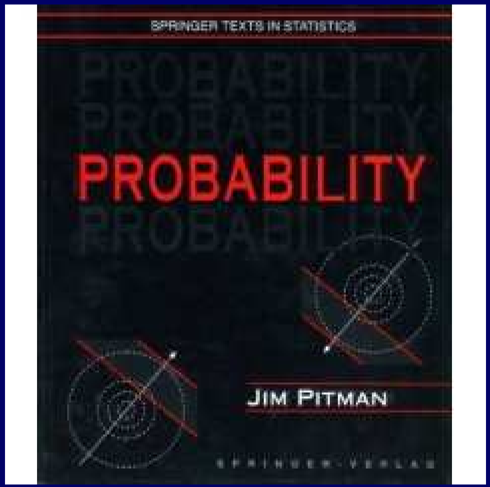
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Section 5.4

Operations on Random Variables

Question: How to compute the distribution of $Z = f(X,Y)$?

Examples: $Z=\min(X,Y)$, $Z = \max(X,Y)$, $Z=X+Y$, $Z = \sqrt{X^2+Y^2}$.

Answer 1: Given the joint density of X and Y, $f(X,Y)$, We can calculate the CDF of Z, $f_Z(z)$ by integrating over the appropriate subsets of the plane.

Recall: If X & Y are independent the joint density is the product of individual densities:

$$f(x,y) = f_X(x)\, f_Y(y),$$

However, in general, it is not enough to know the individual densities.

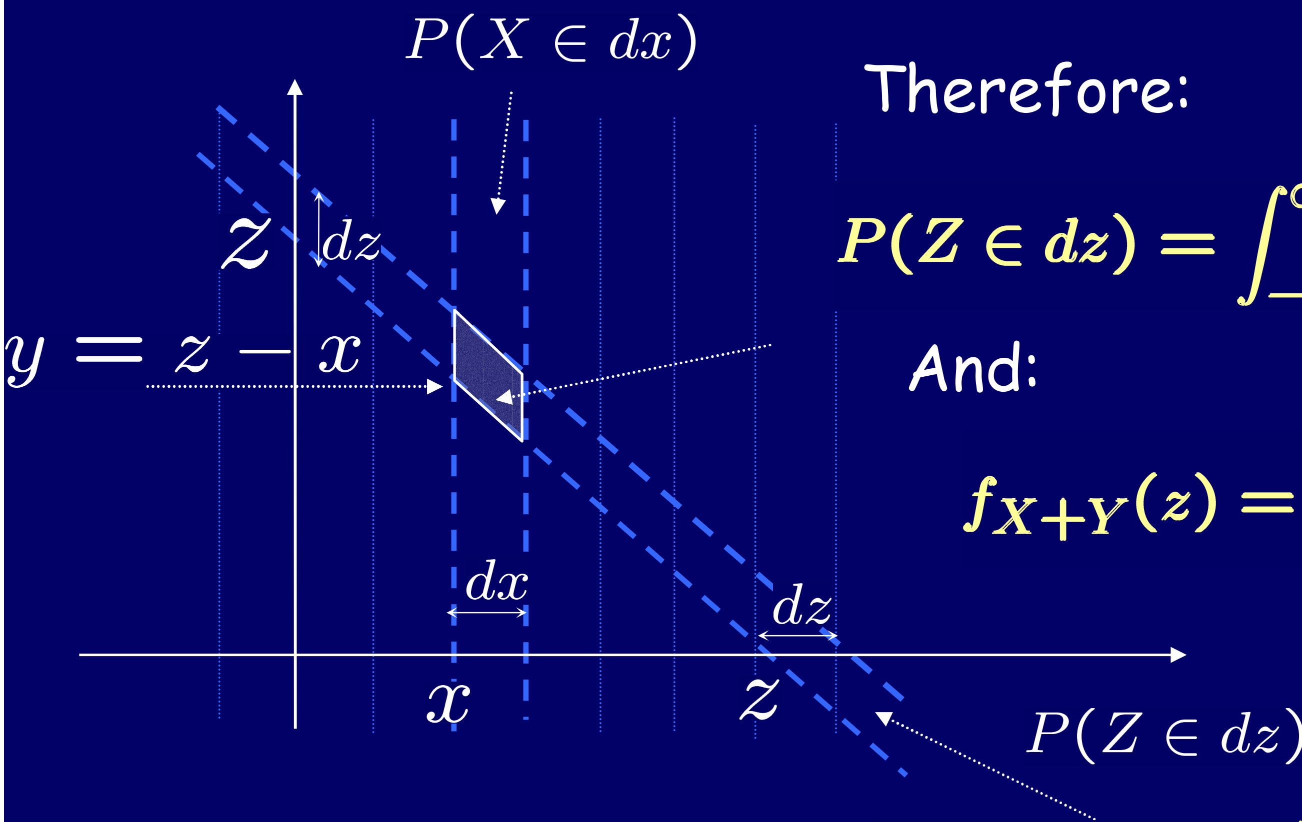
Distribution of Z = X+Y

Discrete case:

$$P(X+Y=z) = \sum_{all\ x} P(X=x, Y=z-x)$$

Continuous case:

$$P(X \in dx, Z \in dz) = f(x, z-x)dxdz$$



Therefore:

$$P(Z \in dz) = \int_{-\infty}^{\infty} f(x, z-x)\, dx\, dz$$

And:

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f(x, z-x)dx$$

For independent X & Y:

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)dx$$

Distribution of Z = X+Y – using CDF

$$F_{X+Y}(a) = P[X+Y \leq a] = \int_{-\infty}^{\infty}\int_{-\infty}^{a-x} f(x,y)\,dy\,dx$$

Therefore

$$f_{X+Y}(a) = \frac{\partial F_{X+Y}(a)}{\partial a} = \int_{-\infty}^{\infty} \frac{\partial \int_{-\infty}^{a-x} f(x,y)\,dy}{\partial a} dx$$

And by the fundemental Theorem of calculus:

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f(x, a-x)\,dx$$

Sum of Independent Exponentials

Suppose that T & U are independent exponential variables with rate λ.

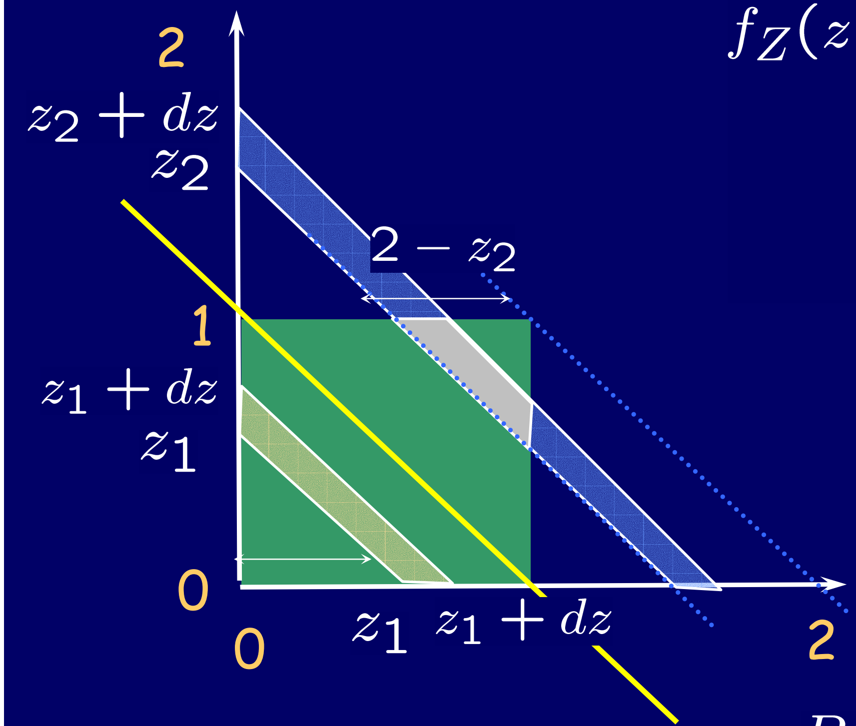
$$f(t,u) = \lambda e^{-\lambda t} \lambda e^{-\lambda u} = \lambda^2 e^{-\lambda(t+u)}, \quad (t,u \geq 0).$$

Let S = T + U, then $f_S(s)$ is given by the convolution formula:

$$\begin{aligned} f_S(s) &= \int_{t \geq 0, t \leq s} \lambda^2 e^{-\lambda t} e^{-\lambda(s-t)} dt \\ &= \lambda^2 e^{-\lambda s} \int_0^s dt \\ &= \lambda^2 e^{-\lambda s} s, \quad (s \geq 0). \end{aligned}$$

Sum of Independent Uniform(0,1)

If X,Y ~ Unif(0,1) and Z = X + Y then



$$f_Z(z) = \int_{-\infty}^{\infty} I_{[0,1]}(x) I_{[0,1]}(z-x)\, dx$$

$$f_Z(z) = \int_{max(z-1,0)}^{min(1,z)} 1 dx$$

$$f_Z(z) = \begin{cases} z, & \text{if } 0 \le z < 1; \\ 2-z, & \text{if } 1 \le z \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

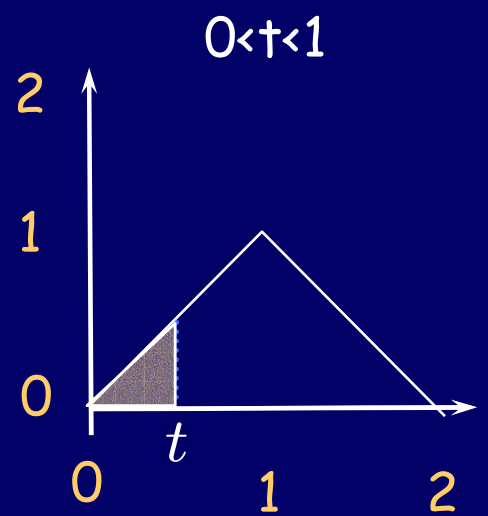
$P(Z \in dz)$

Sum of Independent Uniform(0,1)

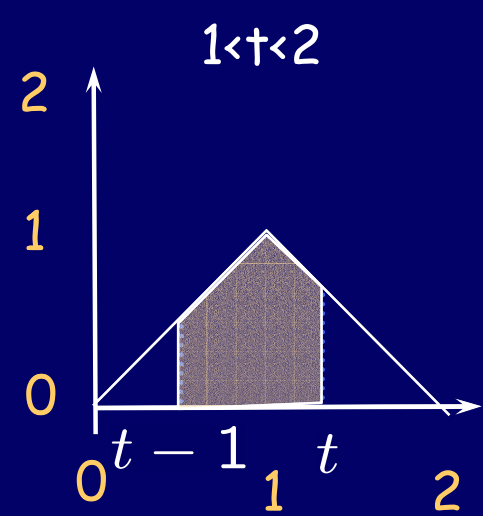
If X,Y,W $\sim$ Unif(0,1) and T = X + Y+W then T = Z + W,

$$f_T(t) = \int_{-\infty}^{\infty} f_Z(z) I_{[0,1]}(t-z)\, dz$$
$$= \int_{t-1}^{t} f_Z(z)\, dz; \qquad (0 < t < 3);$$

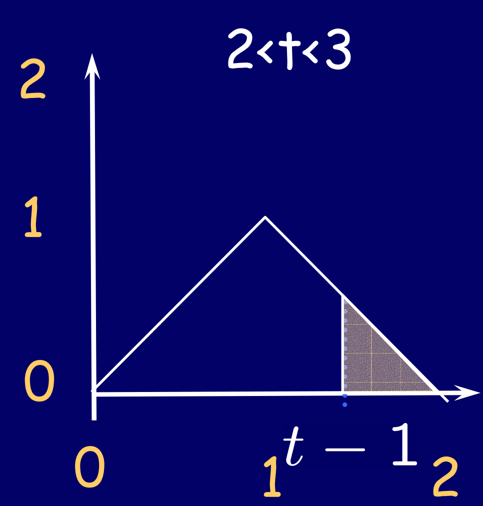
$$f_Z(z) = \begin{cases} z, & \text{if } 0 \le z < 1; \\ 2-z, & \text{if } 1 \le z \le 2; \\ 0, & \text{otherwise.} \end{cases}$$



$$f_T(t) = \frac{1}{2}t^2$$



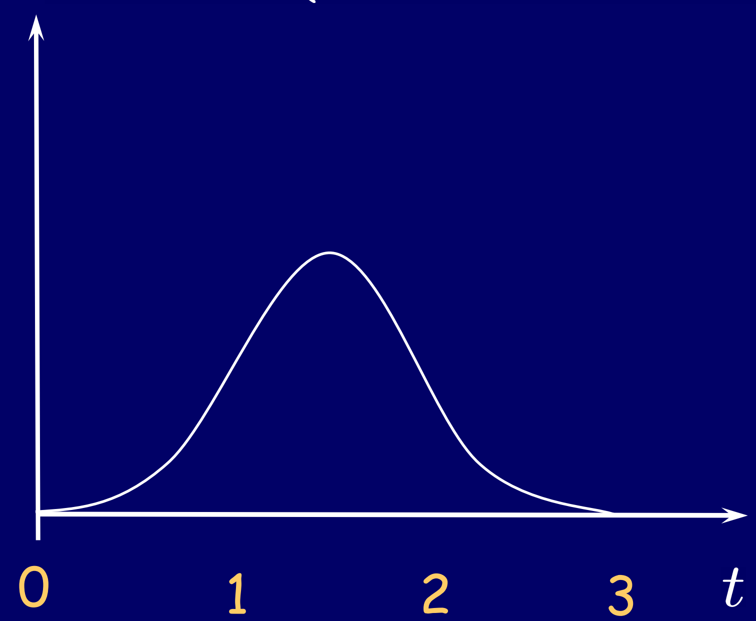
$$f_T(t) = -t^2 + 3t - \frac{3}{2}$$



$$f_T(t) = \frac{1}{2}(3-t)^2$$

Sum of Three Independent Uniform(0,1)

$$f_T(t) = \begin{cases} \frac{1}{2}t^2, \text{ if } 0 \le t < 1; \\ -t^2 + 3t - \frac{3}{2}, \text{ if } 1 \le t \le 2; \\ \frac{1}{2}(3-t)^2, \text{ if } 2 \le t < 3; \\ 0, \text{ otherwise.} \end{cases}$$



The density is symmetric about t=3/2.

Example: Round-off errors

Problem: Suppose three numbers are computed, each with a round-off error Unif$(-10^{-6},10^{-6})$ independently. What is the probability that the sum of the rounded numbers differs from the true sum by more than 2×10^{-6}?

Solution: $x_1 + R_1 \quad x_2 + R_2 \quad x_3 + R_3 \quad R_i \sim \text{Unif}(-10^{-6}, 10^{-6})$

$$x_1 + R_1 + x_2 + R_2 + x_3 + R_3 = x_1 + x_2 + x_3 + (R_1 + R_2 + R_3)$$

Want: $p = 1 - P(-2\times 10^{-6} < R_1 + R_2 + R_3 < 2\times 10^{-6})$

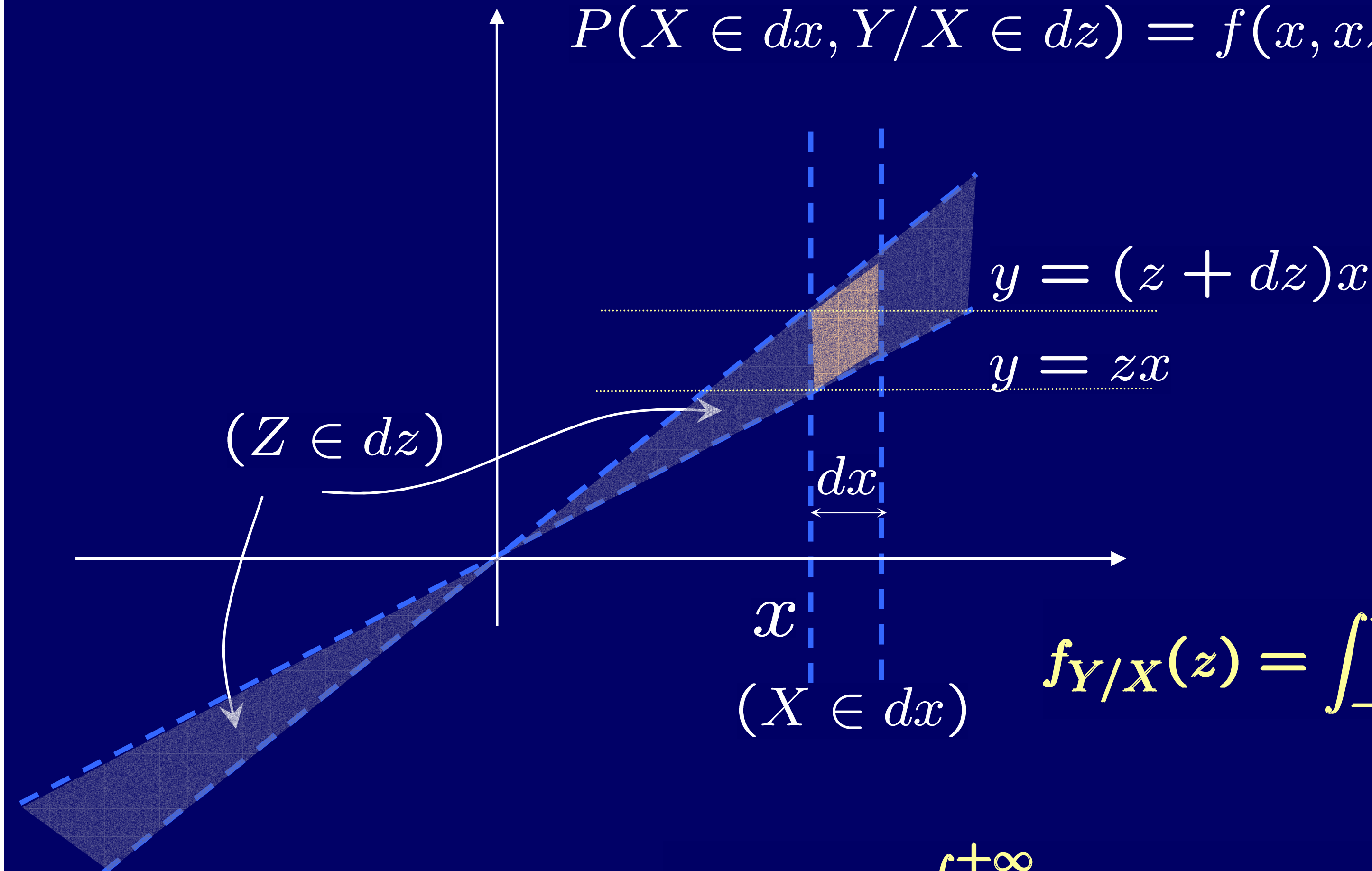
Let: $U_i = (R_i/10^{-6} + 1)/2$. $U_i \sim \text{Unif}(0,1)$ are independent.

$$p = 1 - P(1/2 < U_1 + U_2 + U_3 < 5/2) = 2P(T>5/2) = 2/48.$$

Ratios

Let $Z = Y/X$, then for $z>0$, the event $Z \in dz$ is shaded.

$$P(X \in dx, Y/X \in dz) = f(x, xz)|x|\, dx\, dz$$



$$f_{Y/X}(z) = \int_{-\infty}^{+\infty} |x| f(x, zx)\, dx$$

For independent X & Y: $f_{Y/X}(z) = \int_{0}^{+\infty} |x| f_X(x) f_Y(zx)\, dx \quad (0 < z < \infty).$

Ratio of independent normal variables

Suppose that X & Y~N(0,σ), and independent.

Question: Find the distribution of X/Y.

We may assume that σ =1, since X/Y = X/σ / Y/σ.

$$f_{Y/X}(z) = \int_{-\infty}^{+\infty} |x| \frac{1}{2\pi} e^{-\frac{x^2+x^2z^2}{2}} dx;$$

$$= \int_{0}^{+\infty} x \frac{1}{\pi} e^{-\frac{x^2+x^2z^2}{2}} dx;$$

$$= \left[-\frac{1}{\pi} \frac{1}{z^2+1} e^{-\frac{x^2+x^2z^2}{2}} \right]_0^{+\infty};$$

$$= \frac{1}{\pi} \frac{1}{z^2+1}.$$

This is Cauchy distribution.

Distribution of Z = Y/X – using CDF

Assume X and Y take only positive values. Then

$$F_{Y/X}(a) = P[Y/X \leq a] = \int_{-\infty}^{\infty} \int_{-\infty}^{ax} f(x,y)dydx$$

Therefore

$$f_{Y/X}(a) = \frac{\partial F_{X+Y}(a)}{\partial a} = \int_{-\infty}^{\infty} \frac{\partial \int_{-\infty}^{ax} f(x,y)dy}{\partial a} dx$$

And by the fundemental Theorem of calculus:

$$f_{Y/X}(a) = \int_{-\infty}^{\infty} x f(x, ax)dx$$

Exercise: Calculate the density of $U_1 U_2$ where U_1, U_2 are independent $Unif(0,1)$