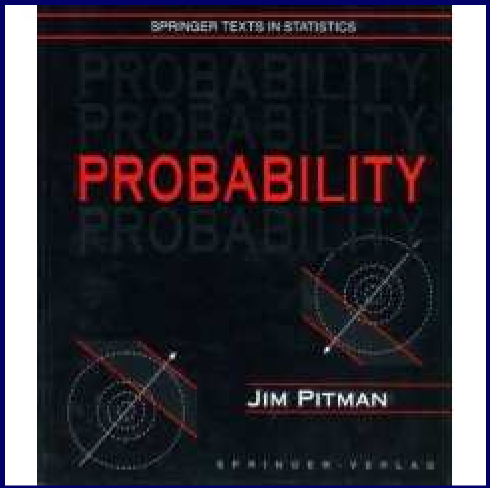
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Section 4.4

Change of Variables

We discuss the following problem:

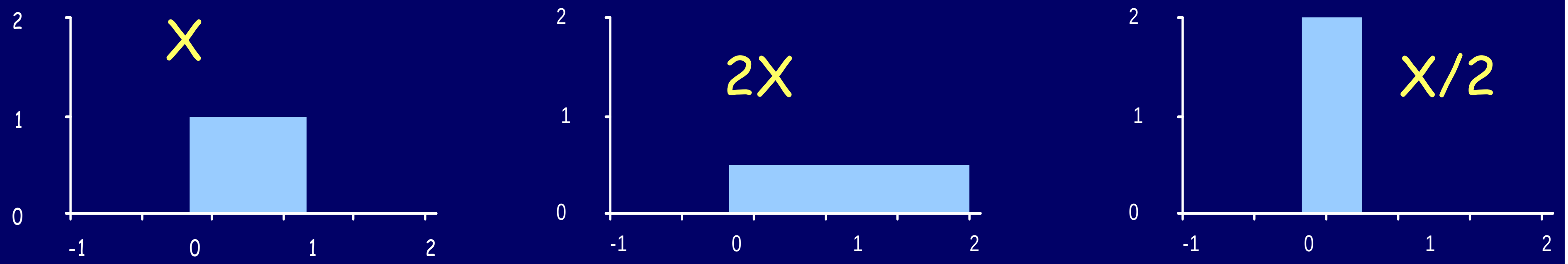
- Suppose a continuous random variable X has a density $f_X(x)$.
- If $Y=g(X)$, what's the density $f_Y(y)$?

Change of Variables

Suppose a continuous random variable X has a density $f_X(x)$. If $Y=g(X)$, what's the density $f_Y(y)$?

<u>Example (Scaling):</u> $X \sim \text{Uniform}(0,1)$ and $g(X) = aX$.

- $P(0<X<1) = 1$ so $P(0<Y/a<1) = 1$ and $P(0<Y<a) = 1$.
- The interval where f_Y is non-zero is of length a,
- The value of f_Y in the interval is 1/a.
- The total area of under f_Y is 1.

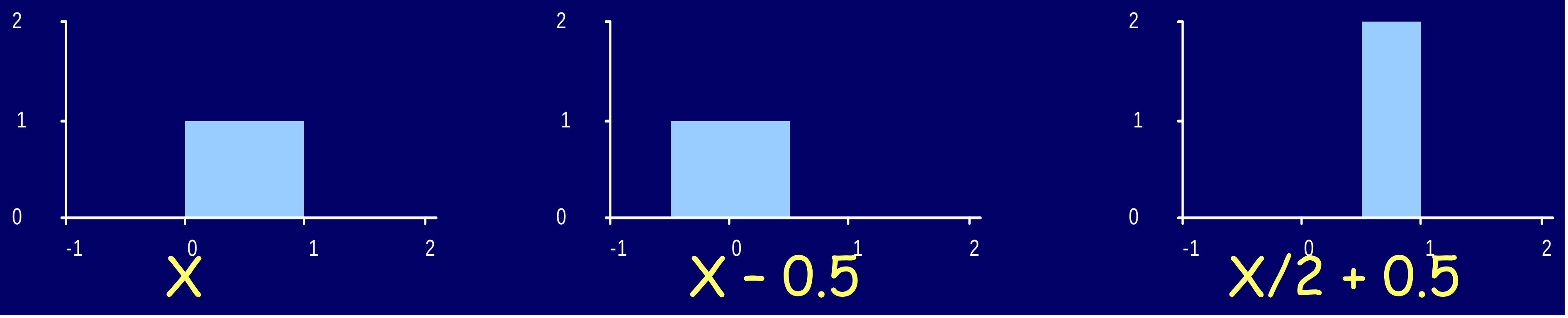


Example: Scaling a uniform

Suppose a continuous random variable X has a density $f_X(x)$. If $Y=g(X)$, what's the density $f_Y(y)$?

Example (Shift): $X\sim$ Uniform(0,1) and $g(X) = X+b$.

- $P(0<X<1) = 1$ so $P(0<Y- b <1) = 1$ and $P(b<Y<1+b) = 1$.
- The length of the interval where f_Y is non-zero is still 1.
- The endpoints of the interval where f_Y is non-zero are shifted by b.
- . The total area under f_Y is 1.



Linear Change of Variables for Densities

Claim: If Y=aX + b, and X has density $f_X(x)$ then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Example: suppose $X \sim N(0,1)$ and Y=aX + b, then

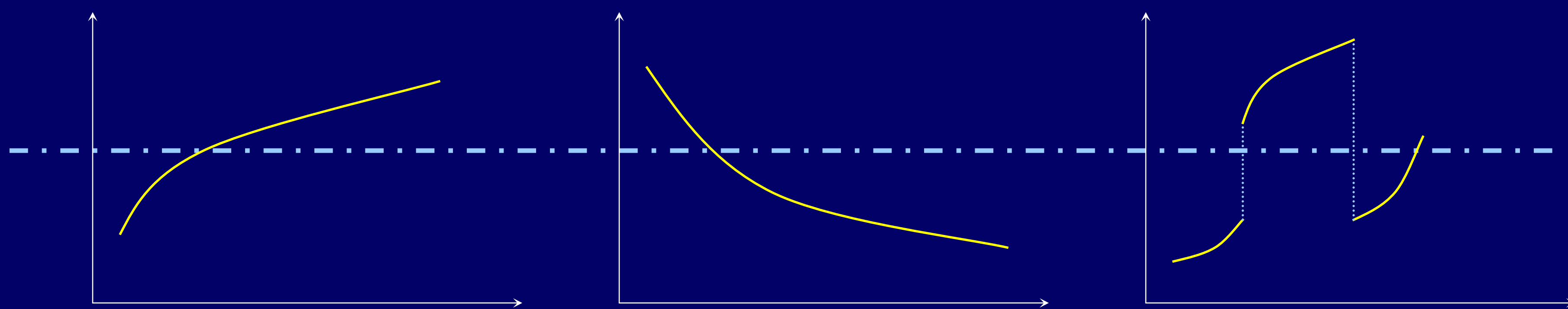
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad f_Y(y) = \frac{1}{|a|\sqrt{2\pi}} e^{-\frac{(y-b)^2}{2a^2}}.$$

This is the calculation of the density of N(a,b) we omitted on section 4.1

1-1 functions

<u>Definition:</u> A function $g(x)$ is 1-1 on an interval (a,b) if for all x,y in (a,b), if $g(x) = g(y)$ then $x = y$.

- In other words, the graph of g cannot cross any horizontal line more than once.



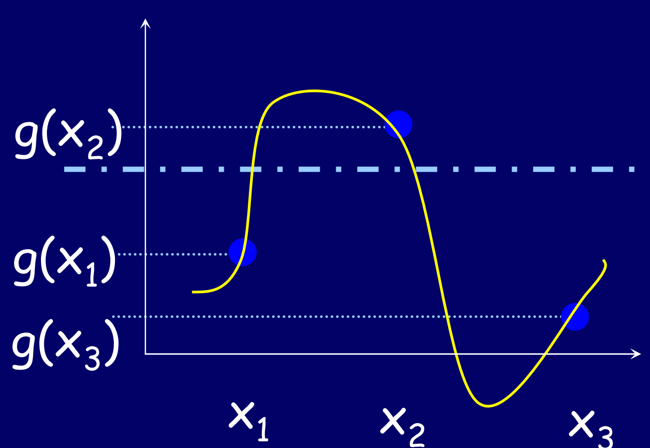
- This implies that g^{-1} is a well defined function on the interval $(g(a), g(b))$ or $(g(b), g(a))$.

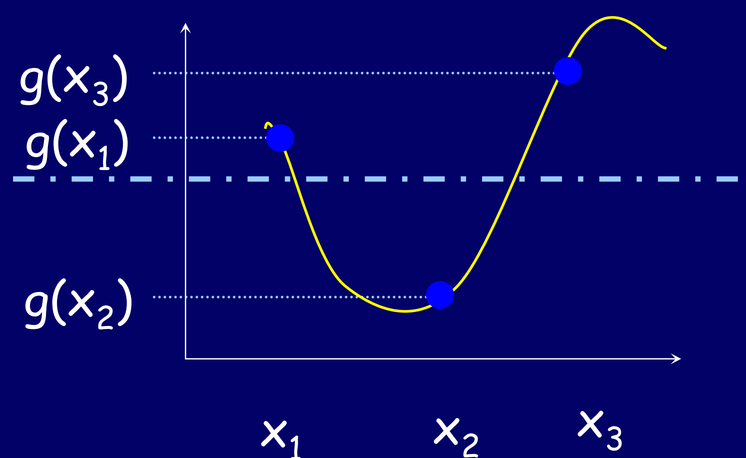
1-1 means Monotonic

<u>Claim:</u> a continuous 1-1 function has to be strictly monotonic, either increasing or decreasing

<u>Pf:</u> If a continuous function $g(x)$ is not strictly monotonic then there exist $x_1 < x_2 < x_3$ such that $g(x_1) \leq g(x_2) \geq g(x_3)$ or $g(x_1) \geq g(x_2) \leq g(x_3)$.

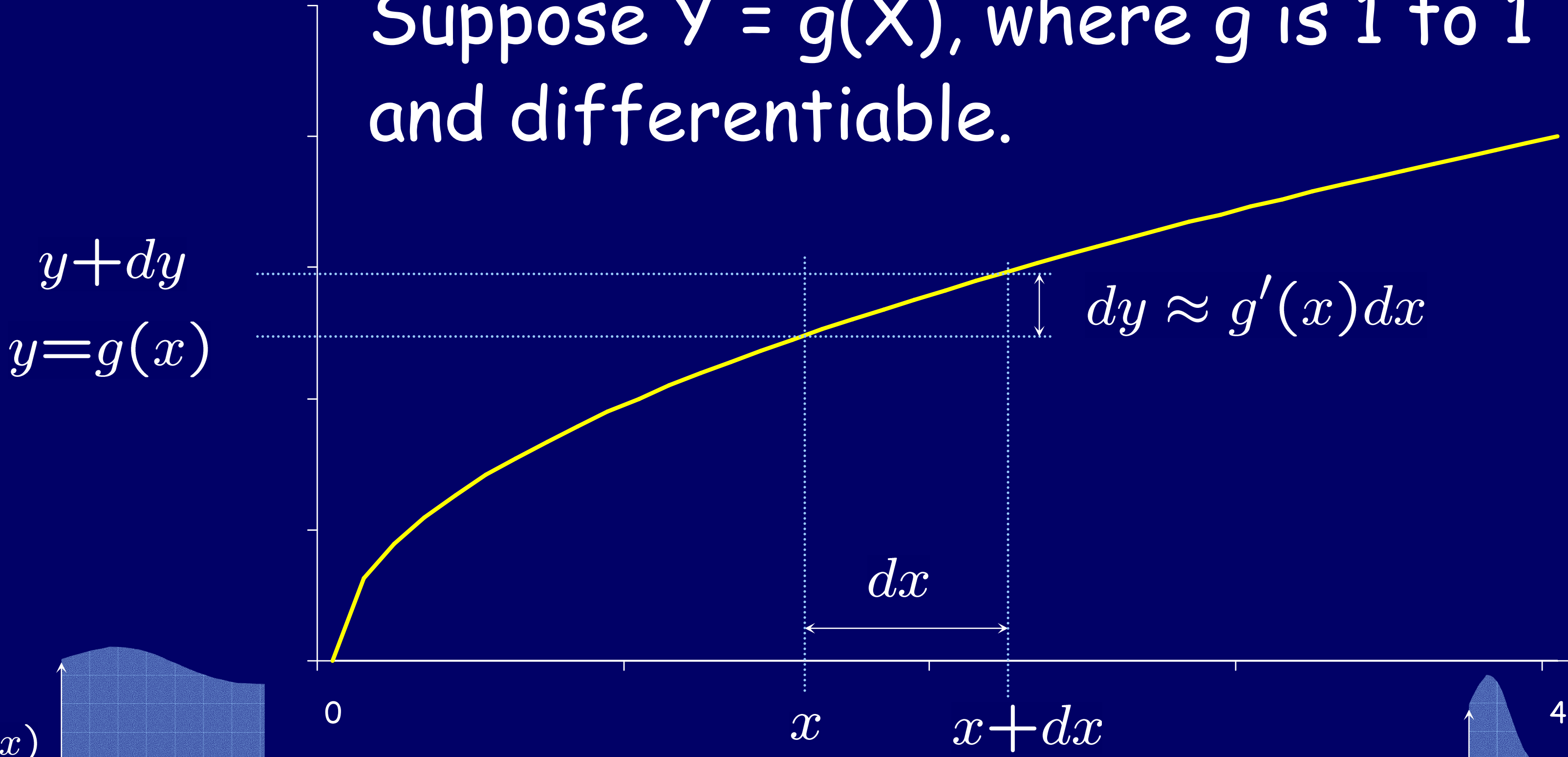
This implies by the mean-value theorem that the function cannot be 1-1.

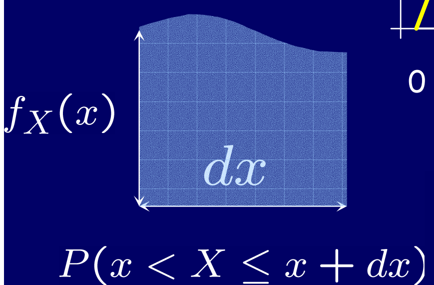




1-1 Differentiable Functions

Suppose Y = g(X), where g is 1 to 1 and differentiable.



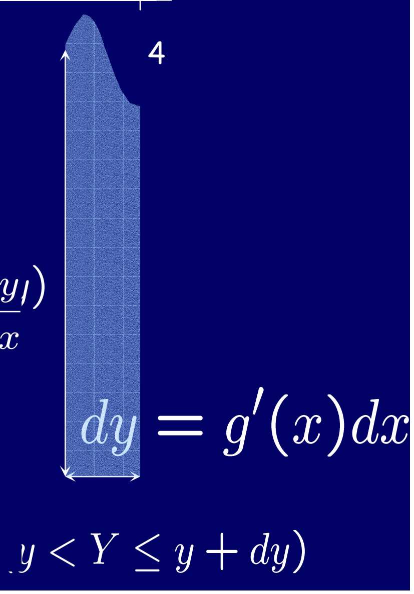


$$f(x) = lim_{dx \to 0} \frac{P[X \in (x, x+dx)]}{dx}$$

$$= lim_{dx \to 0} \frac{P[Y \in (y, y+dy)]}{dx} = lim_{dx \to 0} \frac{P[Y \in (y, y+dy)]}{dy} \frac{dy}{dx})$$

$$= f_Y(y)|g'(x)|dx.$$

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} = \frac{f_X(x)}{|dy/dx|}$$



Change of Variables Formula for 1-1 Differentiable Functions

<u>Claim:</u> Let X be a random variable with density $f_X(x)$ on the range (a,b) . Let Y = g(X) where g is a 1-1 function on (a,b) Then the density of Y on the interval (g(a),g(b)) is:

$$
\begin{aligned}
f_Y(y) &= \frac{f_X(x)}{|g'(x)|} \\
&= \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} \\
&= \frac{f_X(g^{-1}(y))}{|\frac{dy}{dx}|}
\end{aligned}
$$

Exponential function of an Exponential Variable

Example: Let $X \sim \text{Exp}(1)$; $f_X(x) = e^{-x}$, $(x>0)$.

Find the density of $Y = e^{-X}$.

Sol: We have: $dy/dx = -e^{-x} = -y$ and $x = -\log y$.

The range of the new variable is: $e^{-\infty} = 0$ to $e^0 = 1$.

$$f_Y(y) = \frac{f_X(-\log y)}{|-y|} = \frac{e^{\log y}}{y} = 1$$

So $Y \sim \text{Uniform}(0,1)$.

Log of a Uniform Variable

Example: Let $X \sim$ Uniform(0,1) ; $f_X(x) = 1$, $(0<x<1)$.

Find the density of $Y = -1/\lambda \log(X)$, $\lambda > 0$.

Solution: $x = e^{-\lambda y}$ and $dy/dx = -1/(\lambda x) = -e^{\lambda y}/\lambda$,

The range of Y is: 0 to ∞.

$$f_Y(y) = \frac{f_X(e^{-\lambda y})}{e^{-\lambda y}/\lambda} = \frac{1}{e^{\lambda y}/\lambda} = \lambda e^{-\lambda y}$$

So $Y \sim \text{Exp}(\lambda)$.

Square Root of a Uniform Variable

Let $X \sim$ Uniform(0,1) ; $f_X(x) = 1$, $(0<x<1)$.

Find the density of $Y = \sqrt{X}$.

We have: $X = Y^2$ and $dy/dx = 1/(2y)$.

The range of the new variable is: 0 to 1.

$$f_Y(y) = \frac{1}{|1/2y|} = 2y$$

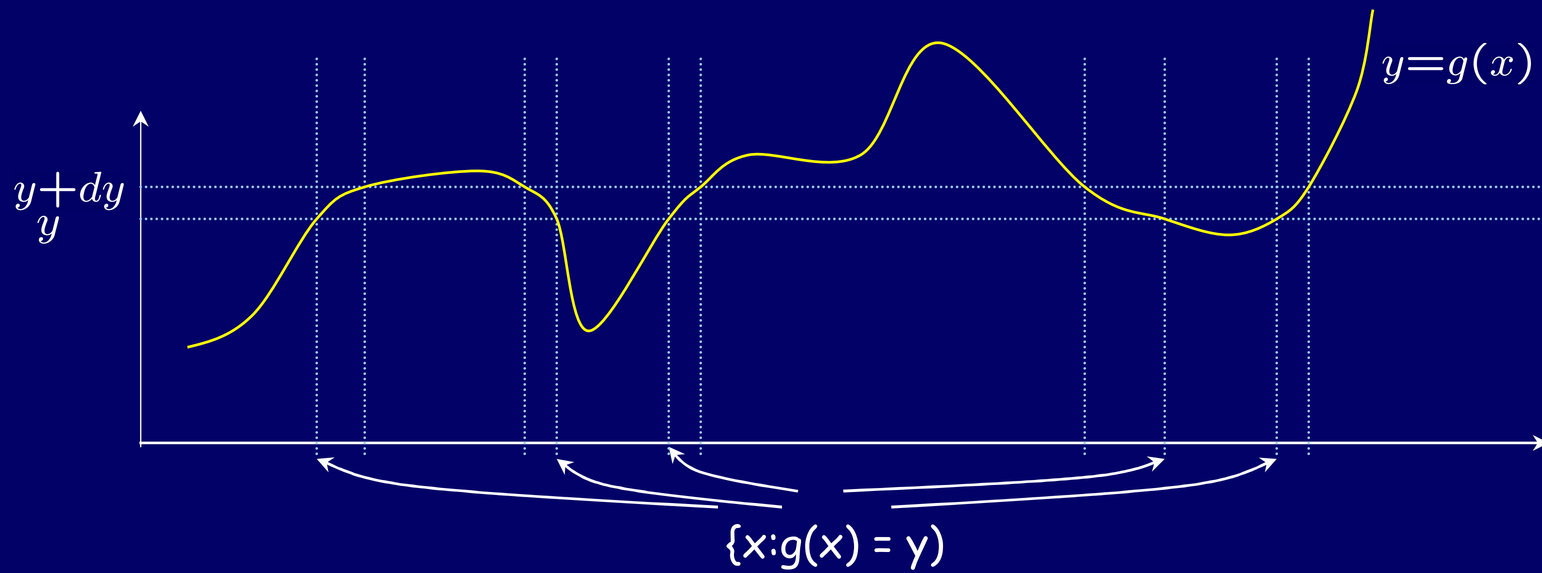
Change of Variables Principle

If X has the same distribution as Y then $g(X)$ has the same distribution as $g(Y)$.

Question: If $X \sim$ Uniform(0,1), what's the distribution of $-(1/\lambda)\log(1-X)$.

Hint: Use the change of variables principle and the result of a previous computation.

Many to One Functions



Now:

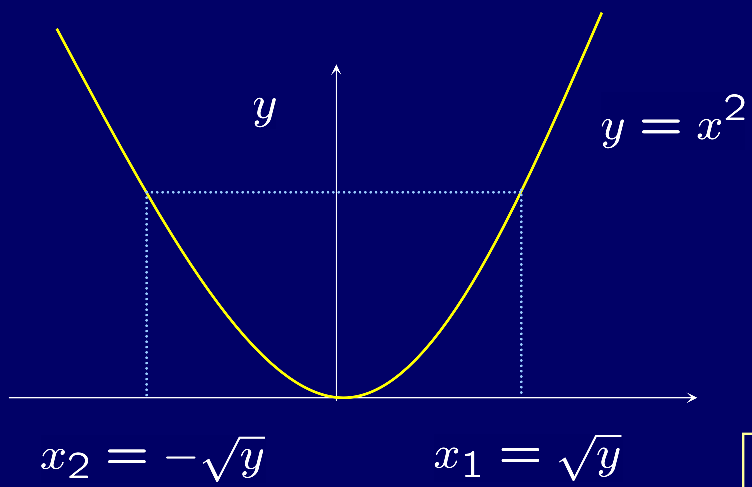
$$P(Y \in (y, y+dy)) = \sum_{x:g(x)=y} P(X \in (x, x+dx))$$

Gives:

$$f_Y(y) = \sum_{x:g(x)=y} f_x(x) / \left|\frac{dy}{dx}\right|$$

Density of the Square Function

Let X have the density $f_X(x)$. Find the density of $Y = X^2$.



$$y = x^2, \quad x = \sqrt{y};$$

$$|dy/dx| = 2|x| = 2\sqrt{y}.$$

$$f_Y(y) = \sum_{x=\pm\sqrt{y}} \frac{f_X(x)}{|2\sqrt{y}|} = \frac{f_X(\sqrt{y}) - f_X(-\sqrt{y})}{2\sqrt{y}}.$$

We saw: If X is the root of a Unif(0,1) RV: $f_X(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$

If $Y=X^2$ then the formula above gives: $f_Y(y) = \frac{2\sqrt{y} - 0}{2\sqrt{y}} = 1.$

And $Y \sim$ Uniform(0,1)

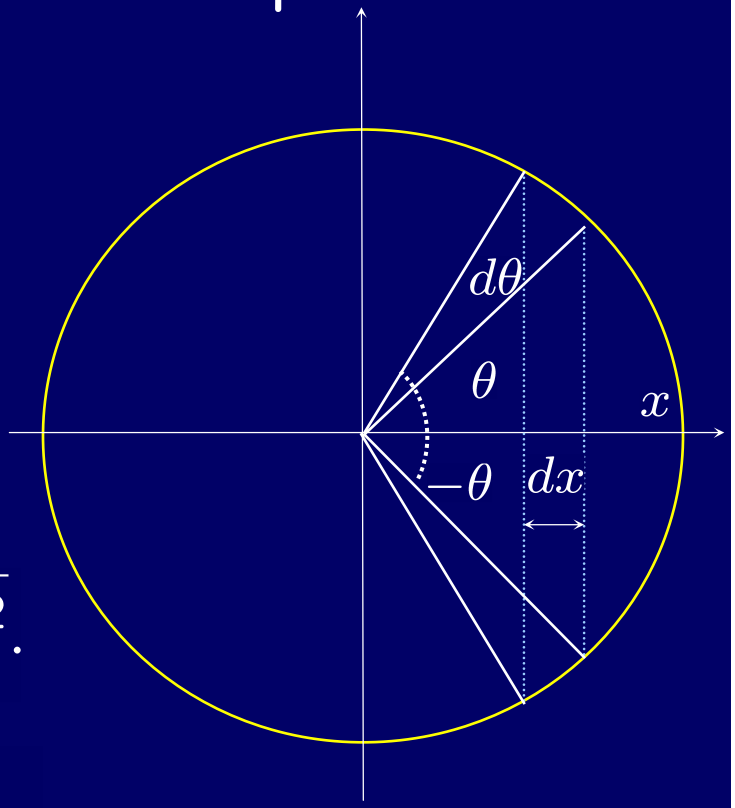
Uniform on a Circle

Problem: Suppose a point is picked uniformly at random from the perimeter of a unit circle.

Find the density of X, the x-coordinate of the point.

Solution: Let Θ be the random angle as seen on the diagram. Then $\Theta \sim$ Uniform $(-\pi,\pi)$. $f_\Theta = 1/(2\pi)$.

$X = \cos(\Theta)$, a 2-1 function on $(-\pi,\pi)$. The range of X is $(-1,1)$.



$$x = \cos\theta, \qquad \left|\frac{dx}{d\theta}\right| = |-\sin\theta| = \sqrt{1-x^2}.$$

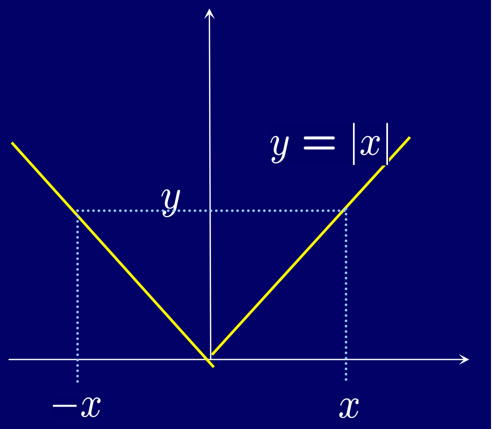
$$f_X(x) = \sum_{\pm\theta} \frac{1}{2\pi\sin\theta} = \frac{1}{\pi\sqrt{1-x^2}}.$$

Uniform on a Circle

Problem: Find E(X).

Solution: Observe that the density $f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}$ is symmetric with respect to 0. So E(X) =0.

Problem: Find the density of Y = |X|.



Solution:

P(Y∈ dy) = 2P(X ∈ dx).

$$f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}.$$

The range of Y is (0,1).

Problem: Find E(Y).

Solution:

$$E(Y) = \frac{2}{\pi}\int_0^1 \frac{y}{\sqrt{1-y^2}}dy = \left[\frac{2}{\pi}\sqrt{1-y^2}\right]_0^1 = \frac{2}{\pi}.$$

Expectation of g(X).

Notice, that is not necessary to find the density of Y = g(X) in order to find E(Y).

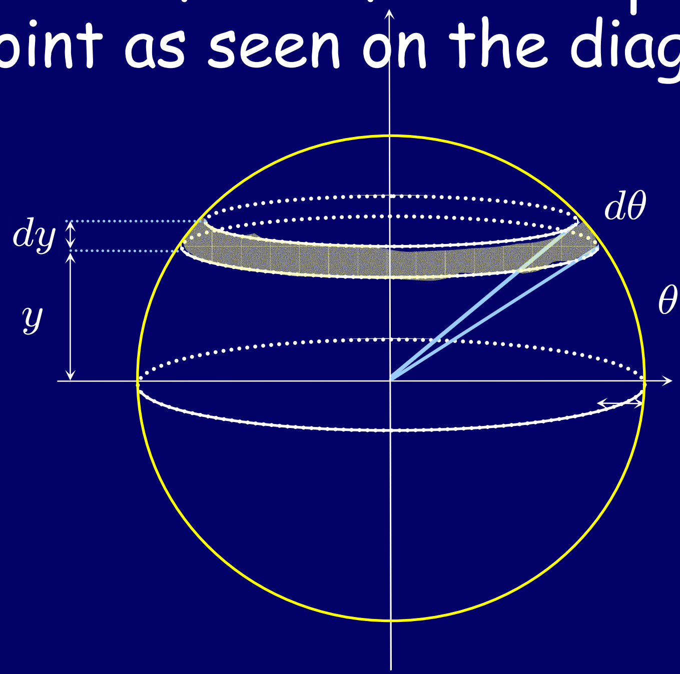
$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

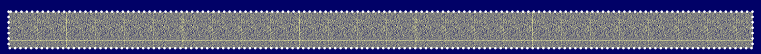
The equality follows by substitution.

$$y = g(x), \quad dy = g'(x) dx.$$

Uniform on a Sphere

<u>Problem:</u> Suppose a point is picked uniformly at random from the surface of a unit sphere. Let Θ be the latitude of this point as seen on the diagram. Find $f_\Theta(\theta)$.





$$\longleftarrow \; 2\pi \cos\theta \; \longrightarrow$$

$$P(\Theta \in d\theta) = \frac{Area\ of\ strip}{Total\ Area} = \frac{2\pi\cos\theta}{4\pi}.$$

$$f_\Theta(\theta) = \frac{\cos\theta}{2}, \qquad \left(-\frac{\pi}{2} < \theta < -\frac{\pi}{2}\right).$$

Let Y be the vertical coordinate of the point. Find $f_Y(y)$.

$$y = \sin\theta, \qquad |\frac{dy}{d\theta}| = |cos\theta|.$$

$$f_Y(y) = \frac{f_\theta(\theta)}{dy/d\theta} = \frac{0.5\cos\theta}{cos\theta} = 0.5.$$

$Y \sim$ Uniform(-1,1).