Introduction to probability

Stat 134      FALL 2005
Berkeley

Lectures prepared by:
Elchanan Mossel
Yelena Shvets

Follows Jim Pitman’s
book:
Probability
Section 4.4
Change of Variables

We discuss the following problem:

• Suppose a continuous random variable $X$ has a density $f_X(x)$.

• If $Y = g(X)$, what’s the density $f_Y(y)$?
Change of Variables

Suppose a continuous random variable $X$ has a density $f_X(x)$. If $Y=g(X)$, what's the density $f_Y(y)$?

Example (Scaling): $X \sim \text{Uniform}(0,1)$ and $g(X) = aX$.

- $P(0<X<1) = 1$ so $P(0<Y/a <1) = 1$ and $P(0<Y<a) = 1$.
- The interval where $f_Y$ is non-zero is of length $a$.
- The value of $f_Y$ in the interval is $1/a$.
- The total area of under $f_Y$ is 1.
Example: Scaling a uniform

Suppose a continuous random variable $X$ has a density $f_X(x)$. If $Y=g(X)$, what’s the density $f_Y(y)$?

Example (Shift): $X \sim \text{Uniform}(0,1)$ and $g(X) = X+b$.
- $P(0<X<1) = 1$ so $P(0<Y-b<1) = 1$ and $P(b<Y<1+b) = 1$.
- The length of the interval where $f_Y$ is non-zero is still 1.
- The endpoints of the interval where $f_Y$ is non-zero are shifted by $b$.
- The total area under $f_Y$ is 1.
Linear Change of Variables for Densities

Claim: If $Y = aX + b$, and $X$ has density $f_X(x)$ then

$$f_Y(y) = \frac{1}{|a|} f_X \left( \frac{y - b}{a} \right).$$

Example: suppose $X \sim N(0,1)$ and $Y = aX + b$, then

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad f_Y(y) = \frac{1}{|a| \sqrt{2\pi}} e^{-\frac{(y-b)^2}{2a^2}}.$$

This is the calculation of the density of $N(a,b)$ we omitted on section 4.1.
1-1 functions

Definition: A function $g(x)$ is 1-1 on an interval $(a,b)$ if for all $x,y$ in $(a,b)$, if $g(x) = g(y)$ then $x = y$.

• In other words, the graph of $g$ cannot cross any horizontal line more than once.

• This implies that $g^{-1}$ is a well defined function on the interval $(g(a), g(b))$ or $(g(b), g(a))$. 
Claim: a continuous 1-1 function has to be strictly monotonic, either increasing or decreasing.

Pf: If a continuous function $g(x)$ is not strictly monotonic then there exist $x_1 < x_2 < x_3$ such that $g(x_1) \leq g(x_2) \geq g(x_3)$ or $g(x_1) \geq g(x_2) \leq g(x_3)$. This implies by the mean-value theorem that the function cannot be 1-1.
1-1 Differentiable Functions

Suppose \( Y = g(X) \), where \( g \) is 1 to 1 and differentiable.

\[
\frac{dy}{dx} \approx g'(x) dx
\]

\[
f(x) = \lim_{dx \to 0} \frac{P[X \in (x, x+dx)]}{dx}
\]

\[
= \lim_{dx \to 0} \frac{P[Y \in (y, y+dy)]}{dy} = \lim_{dx \to 0} \frac{P[Y \in (y, y+dy)]}{dx}
\]

\[
= f_Y(y) |g'(x)| dx.
\]

\[
f_Y(y) = \frac{f_X(x)}{|g'(x)|} = \frac{f_X(x)}{|dy/dx|}
\]

\[
y < Y \leq y + dy
\]
Claim: Let $X$ be a random variable with density $f_X(x)$ on the range $(a,b)$. Let $Y = g(X)$ where $g$ is a 1-1 function on $(a,b)$ Then the density of $Y$ on the interval $(g(a),g(b))$ is:

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}$$

$$= \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

$$= \frac{f_X(g^{-1}(y))}{\left| \frac{dy}{dx} \right|}$$
Exponential function of an Exponential Variable

Example: Let \( X \sim \text{Exp}(1) \); \( f_X(x) = e^{-x}, \ (x>0) \).

Find the density of \( Y = e^{-X} \).

Sol: We have: \( \frac{dy}{dx} = -e^{-x} = -y \) and \( x = -\log y \).

The range of the new variable is: \( e^{-\infty} = 0 \) to \( e^0 = 1 \).

\[
f_Y(y) = \frac{f_X(-\log y)}{|-y|} = \frac{e^{\log y}}{y} = 1
\]

So \( Y \sim \text{Uniform}(0,1) \).
Example: Let $X \sim \text{Uniform}(0,1)$; $f_X(x) = 1$, $(0 < x < 1)$.

Find the density of $Y = -1/\lambda \log(X)$, $\lambda > 0$.

Solution: $x = e^{-\lambda y}$ and $dy/dx = -1/ (\lambda x) = -e^{\lambda y}/\lambda$,

The range of $Y$ is: $0$ to $\infty$.

$$f_Y(y) = \frac{f_X(e^{-\lambda y})}{e^{-\lambda y}/\lambda} = \frac{1}{e^{\lambda y}/\lambda} = \lambda e^{-\lambda y}$$

So $Y \sim \text{Exp}(\lambda)$. 
Square Root of a Uniform Variable

Let $X \sim \text{Uniform}(0,1)$; $f_X(x) = 1, \ (0<x<1)$.

Find the density of $Y = \sqrt{X}$.

We have: $X = Y^2$ and $\frac{dy}{dx} = 1/ (2y)$.

The range of the new variable is: 0 to 1.

$$f_Y(y) = \frac{1}{|1/2y|} = 2y$$
Change of Variables Principle

If $X$ has the same distribution as $Y$ then $g(X)$ has the same distribution as $g(Y)$.

**Question:** If $X \sim \text{Uniform}(0,1)$, what’s the distribution of $-(1/\lambda)\log(1-X)$.

**Hint:** Use the change of variables principle and the result of a previous computation.
Many to One Functions

Now:

\[
P(Y \in (y, y + dy)) = \sum_{x : g(x) = y} P(X \in (x, x + dx))
\]

Gives:

\[
f_Y(y) = \sum_{x : g(x) = y} f_x(x) / \left| \frac{dy}{dx} \right|
\]
Density of the Square Function

Let $X$ have the density $f_X(x)$. Find the density of $Y = X^2$.

$$y = x^2, \quad x = \sqrt{y};$$

$$|dy/dx| = 2|x| = 2\sqrt{y}.$$ 

$$f_Y(y) = \sum_{x = \pm\sqrt{y}} \frac{f_X(x)}{2\sqrt{y}} = \frac{f_X(\sqrt{y}) - f_X(-\sqrt{y})}{2\sqrt{y}}.$$

We saw: If $X$ is the root of a Unif$(0,1)$ RV:

If $Y = X^2$ then the formula above gives:

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

And $Y \sim \text{Uniform}(0,1)$

$$f_Y(y) = \frac{2\sqrt{y} - 0}{2\sqrt{y}} = 1.$$
Uniform on a Circle

Problem: Suppose a point is picked uniformly at random from the perimeter of a unit circle. Find the density of \( X \), the \( x \)-coordinate of the point.

Solution: Let \( \Theta \) be the random angle as seen on the diagram. Then \( \Theta \sim \text{Uniform} (-\pi, \pi) \). \( f_\Theta = 1/(2\pi) \).

\( X = \cos(\Theta) \), a 2-1 function on \((-\pi, \pi)\). The range of \( X \) is \((-1,1)\).

\[
x = \cos \theta, \quad \left| \frac{dx}{d\theta} \right| = \left| -\sin \theta \right| = \sqrt{1 - x^2}.
\]

\[
f_X(x) = \sum_{\pm \theta} \frac{1}{2\pi \sin \theta} = \frac{1}{\pi \sqrt{1 - x^2}}.
\]
Uniform on a Circle

Problem: Find $E(X)$.

Solution: Observe that the density $f_X(x) = \frac{1}{\pi \sqrt{1 - x^2}}$ is symmetric with respect to 0. So $E(X) = 0$.

Problem: Find the density of $Y = |X|$.

Solution: $f_Y(y) = \frac{2}{\pi \sqrt{1 - y^2}}$. $P(Y \in dy) = 2P(X \in dx)$. The range of $Y$ is $(0,1)$.

Problem: Find $E(Y)$.

Solution: $E(Y) = \frac{2}{\pi} \int_0^1 \frac{y}{\sqrt{1 - y^2}} dy = \left[ \frac{2}{\pi} \sqrt{1 - y^2} \right]_0^1 = \frac{2}{\pi}$. 

\[
\begin{align*}
\text{Graph of } y = |x| & \\
\text{Range of } Y & \text{is } (0,1) \text{.}
\end{align*}
\]
Expectation of $g(X)$.

Notice, that is not necessary to find the density of $Y = g(X)$ in order to find $E(Y)$.

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) \, dy = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx.$$  

The equality follows by substitution.

$$y = g(x), \quad dy = g'(x) \, dx.$$
Problem: Suppose a point is picked uniformly at random from the surface of a unit sphere. Let $\Theta$ be the latitude of this point as seen on the diagram. Find $f_\Theta(\theta)$.

Let \( Y \) be the vertical coordinate of the point. Find \( f_Y(y) \).

\[
y = \sin \theta, \quad \left| \frac{dy}{d\theta} \right| = |\cos \theta|.
\]

\[
f_Y(y) = \frac{f_\theta(\theta)}{dy/d\theta} = \frac{0.5 \cos \theta}{\cos \theta} = 0.5.
\]

\[
P(\Theta \in d\theta) = \frac{\text{Area of strip}}{\text{Total Area}} = \frac{2\pi \cos \theta}{4\pi}.
\]

\[
f_\Theta(\theta) = \frac{\cos \theta}{2}, \quad \left( -\frac{\pi}{2} < \theta < -\frac{\pi}{2} \right).
\]

\( Y \sim \text{Uniform}(-1,1) \).