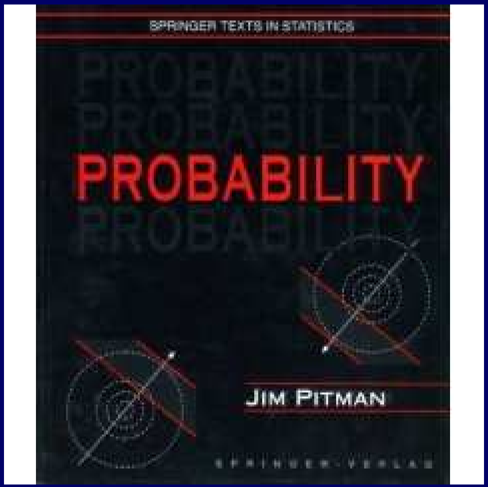
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Section 4.2

Random Times

Random times are often described by a distribution with a density.

Examples:

- Lifetime of an individual from a given population.
- Time until decay of a radioactive atom.
- Lifetime of a circuit.
- Time until a phone call arrives.

Random Times

- If T is a random time then its range is $[0,\infty)$.
- A random time with the density $f(t)$ satisfies:

$$P(a < T \leq b) = \int_a^b f(t)dt\ .$$

- The probability of surviving past time s is called the Survival Function:

$$P(T>s) = \int_s^\infty f(t)dt.$$

- Note that The Survival Function completely characterizes the distribution:

$$P(a < T \leq b) = P(T>a) - P(T>b).$$

Exponential Distribution

Definition: A random time T has an Exponential(λ) Distribution if it has a probability density

$$f(t)=\lambda\, e^{-\lambda t},\ \ (t \top 0).$$

- Equivalently, for ($a \leq b < \infty$)

$$P(a<T<b) = \int_a^b \lambda\, e^{-\lambda t}\, dt = e^{-\lambda a} - e^{-\lambda b}\ .$$

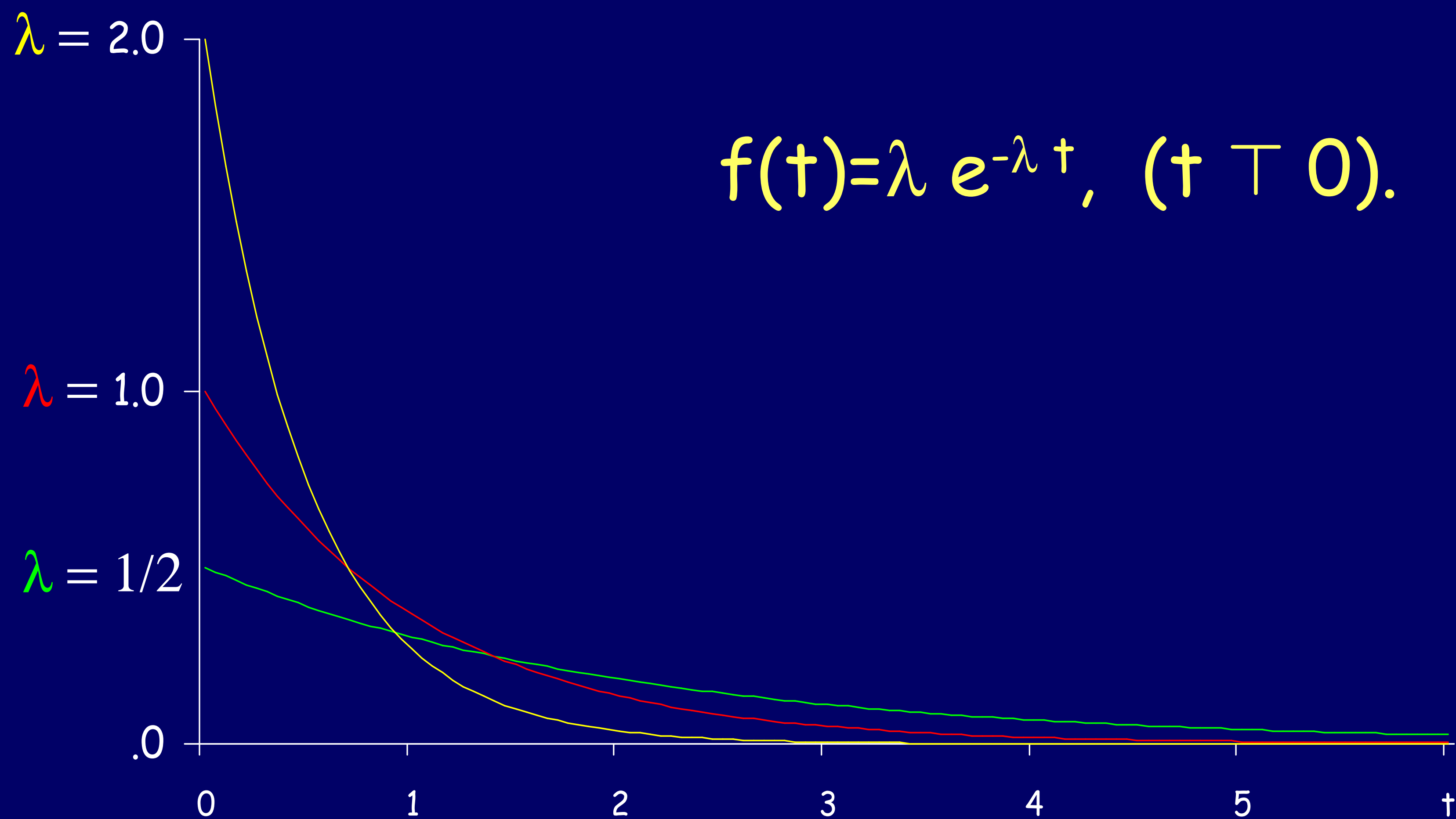
- Equivalently: $P(T>t) = e^{-\lambda t},\ \ (t \top 0).$

Claim: If T has Exp(λ) distribution then:

$$E(T)=SD(T) = 1/\lambda$$

Exponential densities

$f(t)=\lambda\, e^{-\lambda t},\ (t \top 0).$



Memoryless Property

Claim: A positive random variable T has the $\text{Exp}(\lambda)$ distribution iff T has the memoryless property :

$$P(T>t+s \mid T>t) = P(T>s) \ , \ (t,s > 0)$$

• Proof: If $T \sim \text{Exp}(\lambda)$, then

$$P(T>t+s \mid T>t) = P(T>t+s \ \& \ T>t)/P(T>t) = P(T>t+s)/P(T>t)$$

$$= e^{-\lambda(t+s)}/e^{-\lambda t} = e^{-\lambda s} = P(T>s)$$

• Conversely, if T has the memoryless property, the survival function $G(t)=P(T>t)$ must solve $G(t+s) = G(t)G(s)$.

• Thus $L(t) = \log G(t)$ satisfies $L(t+s) = L(t) + L(s)$

and L is decreasing.

• This implies $L(T) = -\lambda t$ or $G(t) = e^{-\lambda t}$

Memoryless Property

Mr. D. is expecting a phone call.

Given that up to the time t minutes the call hasn't come, the chance that the phone call will not arrive in the next s min. is the same as the chance that it hasn't come in the first s min.

$$P(T>t+s \mid T>t) = P(T>s),$$
$$(t,s \geq 0).$$

Hazard Rate

The constant parameter λ can be interpreted as the instantaneous death rate or hazard rate:

$$
\begin{aligned}
P(T \leq t + \Delta \mid T > t) &= 1 - P(T > t + \Delta \mid T > t) \\
&= 1 - P(T > \Delta), \text{ by the memoryless property} \\
&= 1 - e^{-\lambda\Delta} \\
&= 1 - [1 - \lambda \Delta + \tfrac{1}{2} (\lambda \Delta)^2 - \ldots] \\
&\approx \lambda \Delta
\end{aligned}
$$

The approximation holds if Δ is small (so we can neglect higher order terms).

Exponential & Geometric Distributions

Claim:

Suppose $G(p)$ has a geometric p distribution, then for all t and λ: $\lim_{p \to 0} P(pG(p)/\lambda > t) = e^{-\lambda t}$.

Proof:

- $\lim_{p \to 0} P(p\, G(p)/\lambda > t) = \lim_{p \to 0} P(G(p) > \lambda t/p)$

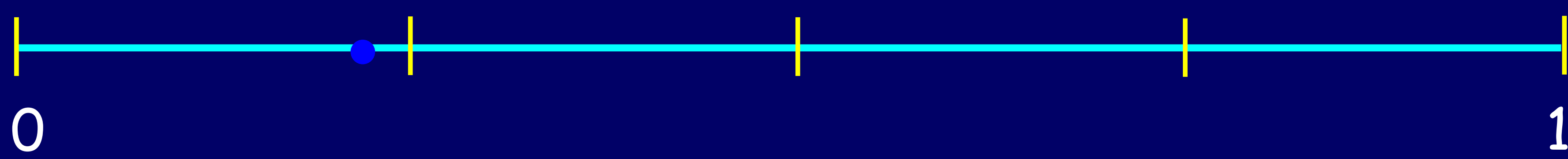
$$= \lim_{p \to 0} (1-p)^{\lfloor \lambda t/p \rfloor}$$

$$= \lim_{p \to 0} (1-p)^{\lambda t/p} = e^{-\lambda t}.$$

Relationship to the Poisson Process

Let the unit interval be subdivided into n cells of length 1/n each occupied with probability

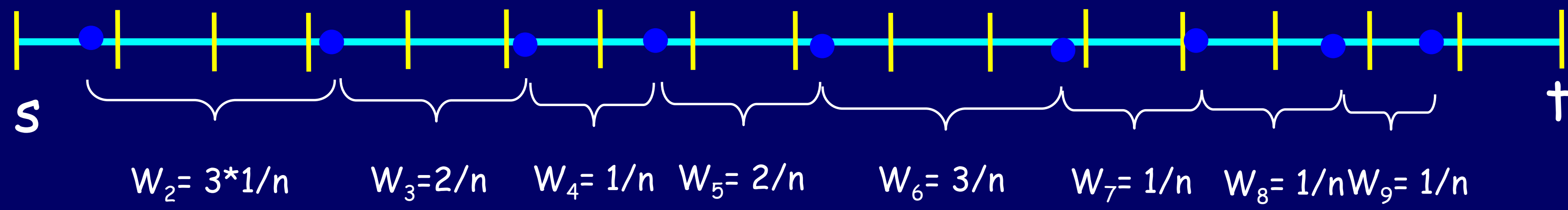
$p = \lambda / n$.



Note that the number of occupied cells is $\mathrm{Bin}(n, \lambda/n)$.

Relationship to the Poisson Process

Question: What's the distribution of the successes in an interval (s,t)?



Number of successes in (t-s) n trials is distributed as $\text{Bin}((t-s)\,n,\lambda/n) \approx \text{Poisson}((t-s)\lambda)$.

Question: What is the distribution of W_i's, the waiting times between success i-1 and success i.

$$\text{Time} = (\#\text{ of empty cells}+1)^* \; 1/n \sim$$

$$\sim \text{Geom}(\lambda/n)^* 1/n \approx \exp(\lambda).$$

Two descriptions of Poisson Arrival Process

1. Counts of arrivals:

- The Number of arrivals in a *fixed* time interval of length t is a Poisson(λt) random variable:
  - $P(N(t) = k) = e^{-\lambda t} (\lambda t)^{-k}/k!$
- # of arrivals on disjoint intervals are independent.

2. Times between successes:

- Waiting times W_i between successes are Exp(λ) random variables: $P(W_i>t) = e^{-\lambda t}$
- The random variables W_i are independent.

Poisson Arrival Process

- Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

- So the number of calls $N(s,t)$ between the minutes s and t satisfies:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

- The waiting time between the $(i-1)^{st}$ and the i^{th} calls satisfies:

$$W_i \sim \text{Exp}(3).$$

Question: What's the probability that no call arrives between $t=0$ and $t=2$?

Solution: $N(0,2) \sim \text{Poisson}(6)$, $P(N(0,2) = 0) = e^{-6} = 0.0025$.

Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes s and t, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the i^{th} calls, we have:

$$W_i \sim \text{Exp}(3).$$

Question: What's the probability that the first call after $t=0$ takes more than 2 minute to arrive?

Solution: $W_1 \sim \text{Exp}(3)$, $P(W_1 \geq 2) = e^{-3*2} = 0.0025.$

Note: the answer is the same as in the first question.

Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes s and t, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the i^{th} calls, we have:

$$W_i \sim \text{Exp}(3).$$

Question: What's the probability that no call arrives between $t=0$ and $t=2$ and at most 4 calls arrive between $t=2$ and $t=3$?

Solution: By independence of $N(0,2)$ and $N(2,3)$, this is

$P(N(0,2) = 0) * P(N(2,3) \leq 4)$

$= e^{-6} e^{-3} (1 + 3 + 3^2/2! + 3^3/3! + 3^4/4!) = 0.0020,$

Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes s and t, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the i^{th} calls, we have:

$$W_i \sim \text{Exp}(3).$$

Question: What's the probability that the fourth call arrives within 30 seconds of the third?

Solution: $P(W_4 \le 0.5) = 1 - P(W_4 > 0.5) = 1 - e^{-3/2} = 0.7769.$

Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes s and t, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the i^{th} calls, we have:

$$W_i \sim \text{Exp}(3).$$

Question: What's the probability that the fifth call takes more than 2 minutes to arrive?

Solution: $P(W_1 + W_2 + W_3 + W_4 + W_5 > 2) = P(N(0,2) \leq 4)$

$$P(N(0,2) \leq 4) = e^{-6}(1 + 6 + 6^2/2! + 6^3/3! + 6^4/4!) = 0.2851.$$

Poisson r^{th} Arrival Times

- Let T_r denote the time of the r^{th} arrival in a Poisson process with intensity λ. The distribution of T_r is called Gamma(r,λ) distribution.

- Note that: $T_r = W_1 + W_2 + \ldots + W_r$, where W_i's are the waiting times between arrivals.

- Density: $P(T_r \in (t,t+dt))/dt = P[N(0,t) = r-1] * (\text{Rate});$

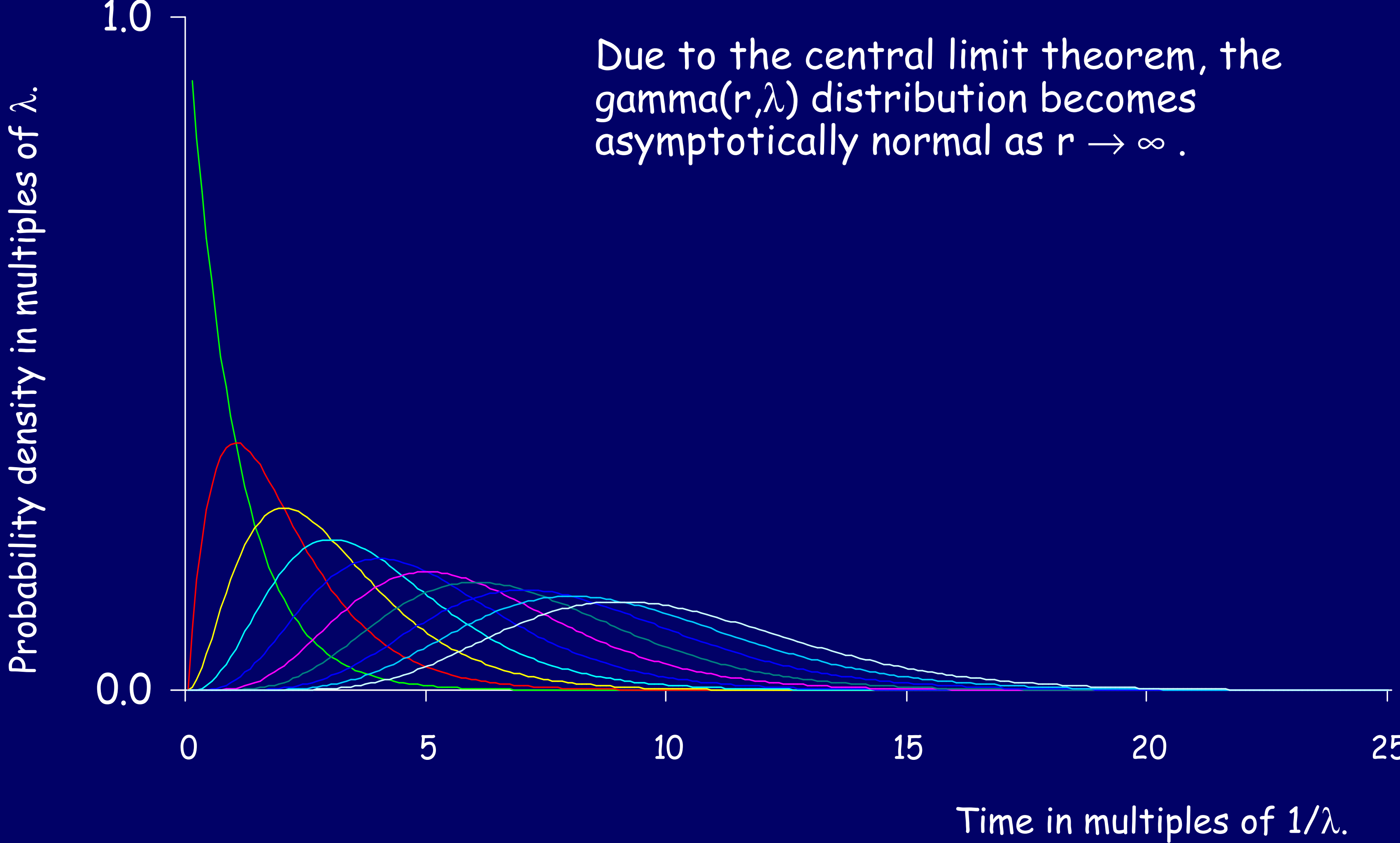
$$P(T_r \in (t, t+dt)) = e^{-\lambda t}\frac{(\lambda t)^{r-1}}{(r-1)!}\lambda dt$$

- Right tail probability: $P(T_r > dt) = P(N(0,t) \leq r-1);$

$$P(T_r > t) = \sum_{k=0}^{r-1} e^{-\lambda t}\frac{(\lambda t)^k}{(k)!}.$$

- Mean and SD: $E(T_r) = r/\lambda \quad SD(T_r) = \sqrt{r}/\lambda.$

Gamma Densities for r=1 to 10.



Due to the central limit theorem, the gamma(r,λ) distribution becomes asymptotically normal as $r \to \infty$.

Relationship between Poisson and Exponential.

Property/idea	Discrete	Continuous
Wait time until one success	Geometric (p) (trials)	Exponential (λ) (time)
Number of successes in a fixed interval	Binomial (n, p) (fixed trials n)	Poisson (λt) (fixed time t)
Wait time until k successes	NegBin (k, λ)	Gamma (k, λ)
P(kth success/hit is after some time)	NegBin (k, λ) > n	Gamma (k, λ) > t
P(less than k successes in some time)	Binomial (n, p) ≤ k-1	Poisson (λt) ≤ k-1