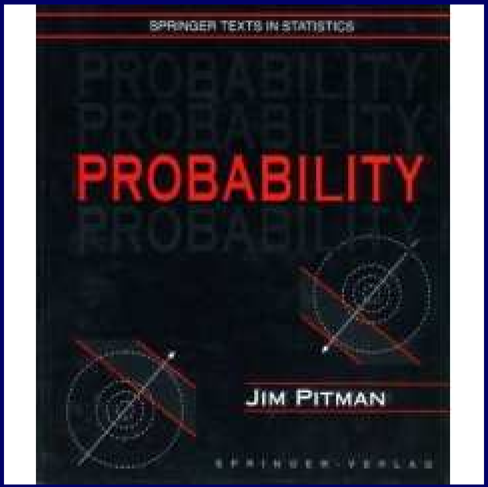
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Section 3.5

The Poisson (μ) Distribution

The Poisson distribution with parameter μ or Poisson(μ) distribution is the distribution of probabilities $P_\mu(k)$ over {0,1,2,...} defined by:

$$P(k) = e^{-\mu} \frac{\mu^k}{k!}$$

Poisson Distribution

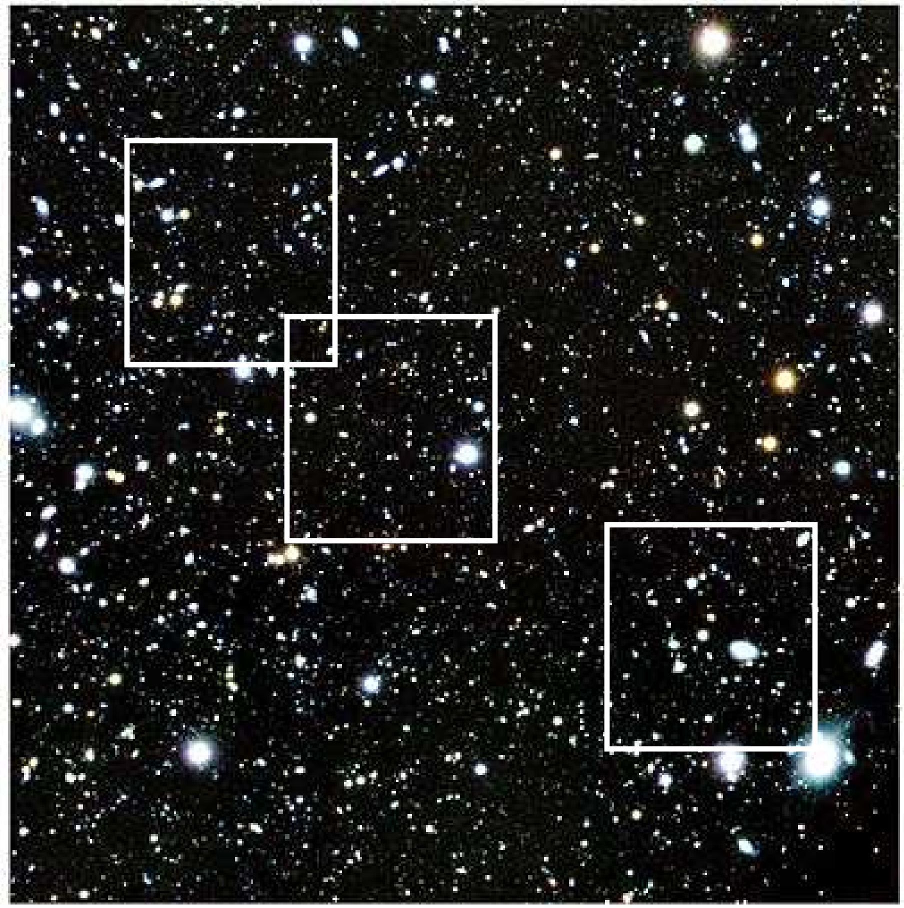
- Recall that Poisson(μ) is a good approximation for $N = Bin(n,p)$ where
- n is large
- p is small and
- $\mu = np$.

Poisson Distribution

<u>Example:</u>

$N_{galaxies}$:

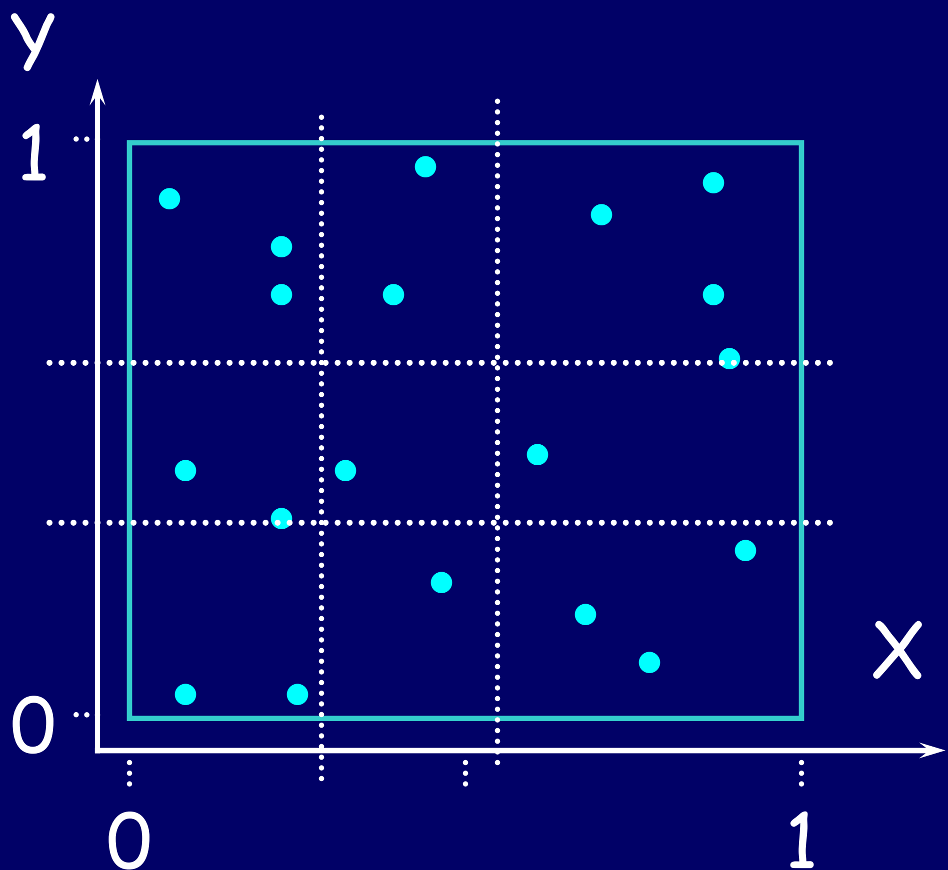
the number of super-nova found in a square inch of the sky is distributed according to the Poisson distribution.



Colour-Composite of the Sky Field With Several High-Redshift Galaxies
VLT YEPUN + FORS2

ESO PR Photo 25a/03 (21 August 2003) © European Southern Observatory

Poisson Distribution



Example:

N_{drops}: the number of rain drops falling on a given square inch of the roof in a given period of time has

a Poisson distribution.

Poisson Distribution: μ and σ

Since N = Poisson(μ) $\sim$ Bin(n,μ/n) we expect that:

$$E(N) = \lim_{np=\mu, p\to 0, n\to\infty} (np) = \mu;$$

$$SD(N) = \lim_{np=\mu, p\to 0, n\to\infty} \sqrt{np(1-p)} = \sqrt{\mu}.$$

Next we will verify it.

Mean of the Poisson Distribution

By direct computation:

$$
\begin{aligned}
E[N] &= \sum_{k=0}^{\infty} kP[N=k] = \sum_{k=1}^{\infty} ke^{-\mu}\frac{\mu^k}{k!} \\
&= e^{-\mu}\mu \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!} = e^{-\mu}\mu \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \\
&= e^{-\mu}\mu e^{\mu} = \mu.
\end{aligned}
$$

SD of the Poisson Distribution

$$
\begin{aligned}
E[N(N-1)] &= \sum_{k=0}^{\infty} k(k-1)P[N=k] = \sum_{k=2}^{\infty} k(k-1)e^{-\mu}\frac{\mu^k}{k!} \\
&= e^{-\mu}\mu^2 \sum_{k=2}^{\infty} \frac{\mu^{k-2}}{(k-2)!} = e^{-\mu}\mu^2 \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \\
&= e^{-\mu}\mu^2 e^{\mu} = \mu^2.
\end{aligned}
$$

So

$$E[N^2] = E[N(N-1)] + E[N] = \mu^2 + \mu$$

$$Var[N] = E[N^2] - E[N]^2 = (\mu^2 + \mu) - \mu^2 = \mu.$$

$$SD(N) = \sqrt{\mu}.$$

Sum of independent Poissons

- Consider two independent r.v.'s :

$N_1 \sim \text{Poisson}(\mu_1)$ and $N_2 \sim \text{Poisson}(\mu_2)$.

- We approximately have:

$N_1 \sim \text{Bin}(\mu_1/p, p)$ and $N_2 \sim \text{Bin}(\mu_2/p, p)$,

where p is small.

- So if we pretend that $\mu_1/p, \mu_2/p$ are integers then:

$N_1 + N_2 \sim \text{Bin}((\mu_1+\mu_2)/p, p) \sim \text{Poisson}(\mu_1+\mu_2)$

Sum of independent Poissons

- Claim: if $N_i \sim \text{Poisson}(\mu_i)$ are independent then $N_1 + N_2 + \ldots + N_r \sim \text{Poisson}(\mu_1 + \ldots \mu_r)$
- Proof (for r=2):

$$
\begin{aligned}
P[N_1 + N_2 = k] &= \sum_{j=0}^{k} P[N_1 = j]P[N_2 = k-j] \\
&= \sum_{j=0}^{k} e^{-\mu_1}\frac{\mu_1^j}{j!} e^{-\mu_2}\frac{\mu_2^{k-j}}{(k-j)!} \\
&= \frac{e^{-(\mu_1+\mu_2)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \mu_1^j \mu_2^{k-j} \\
&= \frac{(\mu_1 + \mu_2)^k e^{-(\mu_1+\mu_2)}}{k!}
\end{aligned}
$$

Poisson Scatter

The Poisson Scatter Theorem: Considers particles hitting the square where:

- Different particles hit different points and
- If we partition the square into n equi-area squares then each square is hit with the same probability p_n independently of the hits of all other squares

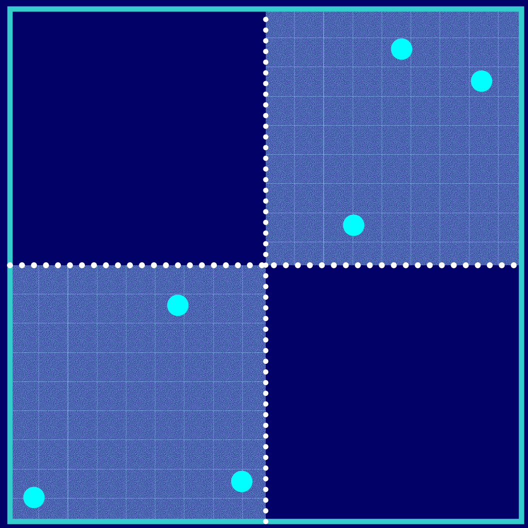
Poisson Scatter

<u>The Poisson Scatter Theorem:</u> states that under these two assumptions there exists a number $\lambda > 0$ s.t:

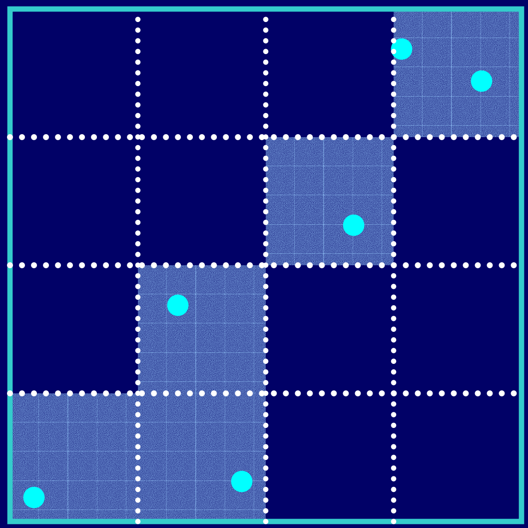
- For each set B, the number N(B) of hits in B satisfies $N(B) \sim Poisson(\lambda \times Area(B))$
- For disjoint sets $B_1,\ldots,B_r$, the numbers of hits $N(B_1), \ldots, N(B_r)$ are independent.

- The process defined by the theorem is called Poisson scatter with intensity λ

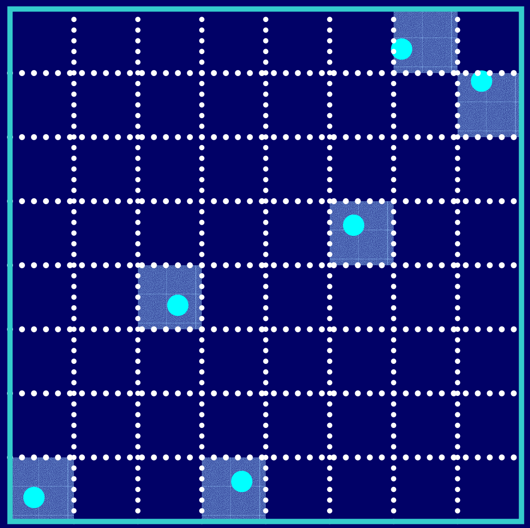
Poisson Scatter



Poisson Scatter



Poisson Scatter



Poisson Scatter Thinning

Claim: Suppose that in a Poisson scatter with intensity λ

- each point is kept with prob. p and erased with probability 1-p
- independently of its position and the erasure of all other points.

- Then the scatter of points kept is a Poisson scatter with intensity p λ.