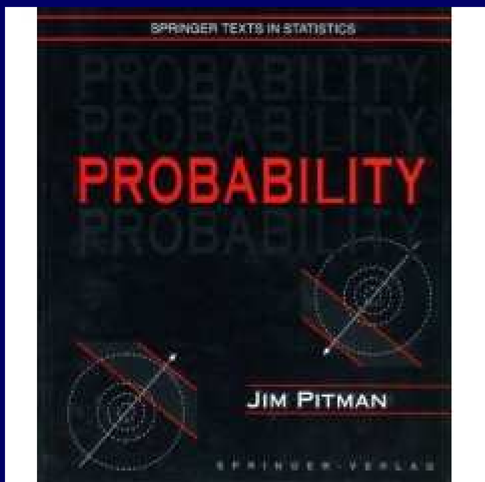
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Section 3.2

Mean of a Distribution

- The mean μ of a probability distribution $P(x)$ over a finite set of numbers x is defined
- The mean is the average of the these numbers weighted by their probabilities:

$$\mu = \sum_x x \, P(X=x)$$

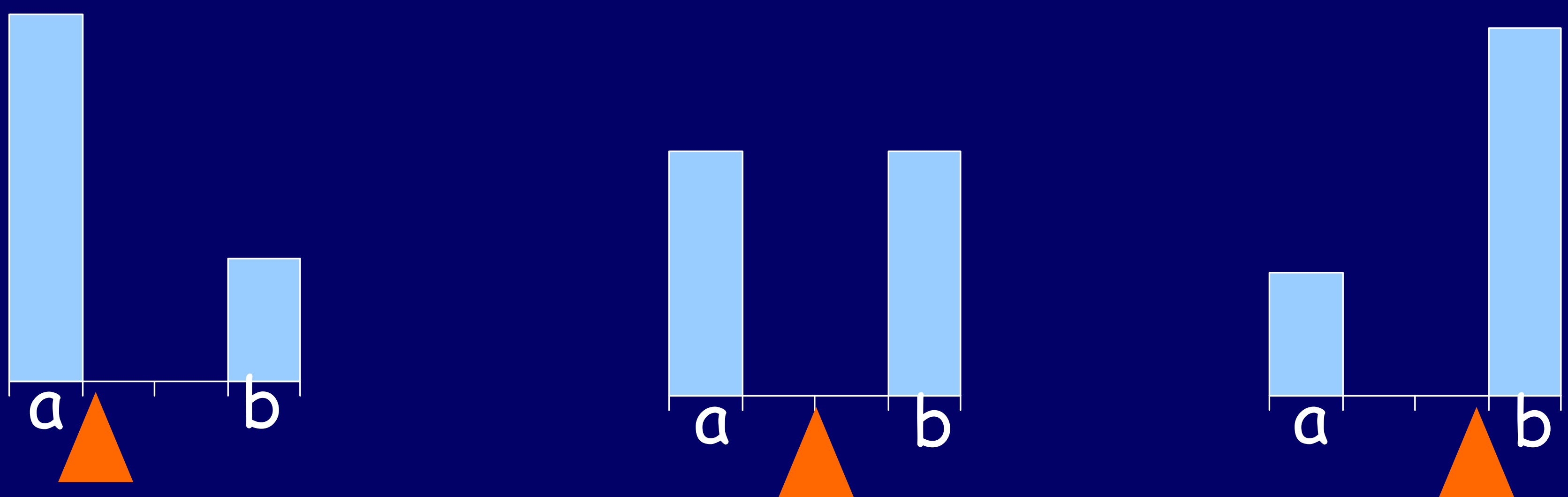
Expectation

The expectation (also expected value or mean) of a random variable X is the mean of the distribution of X.

$$E(X) = \sum_x x\, P(X=x)$$

Two Value Distributions

- If X is a Bernoulli(p) variable over {a,b}, then $E(X) = pa + (1-p)b$.
- If we think of p and q as two masses sitting over a and b then E(X) would correspond to the point of balance:



A Fair Die

Let X be the number rolled with a fair die.

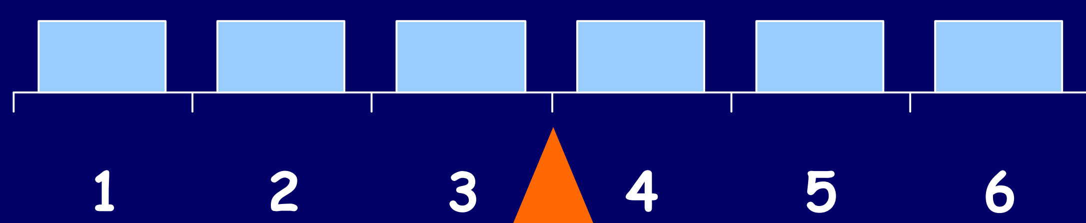
Question: What is the expected value of X?

We can compute $E(X)$ by definition:

$$E(X) = 1*1/6 + 1*1/6 + 3*1/6 + 4*1/6 + 5*1/6 + 6*1/6.$$

$$= 3.5$$

Alternatively, we could find the point of balance on the histogram:



Binomial(10, ½)

<u>Question:</u> Let Z be a variable with a binomial(10, ½) distribution.

What is E[Z]?

By definition:

$$E(Z) = \sum_{i=0}^{10} iP(Z=i)$$

Binomial(10, ½)

$$E(Z) = 0 + 1*10\left(\frac{1}{2}\right)^{10} + 2*45\left(\frac{1}{2}\right)^{10} + 3*120\left(\frac{1}{2}\right)^{10} + 4*210\left(\frac{1}{2}\right)^{10} + 5*252\left(\frac{1}{2}\right)^{10}$$

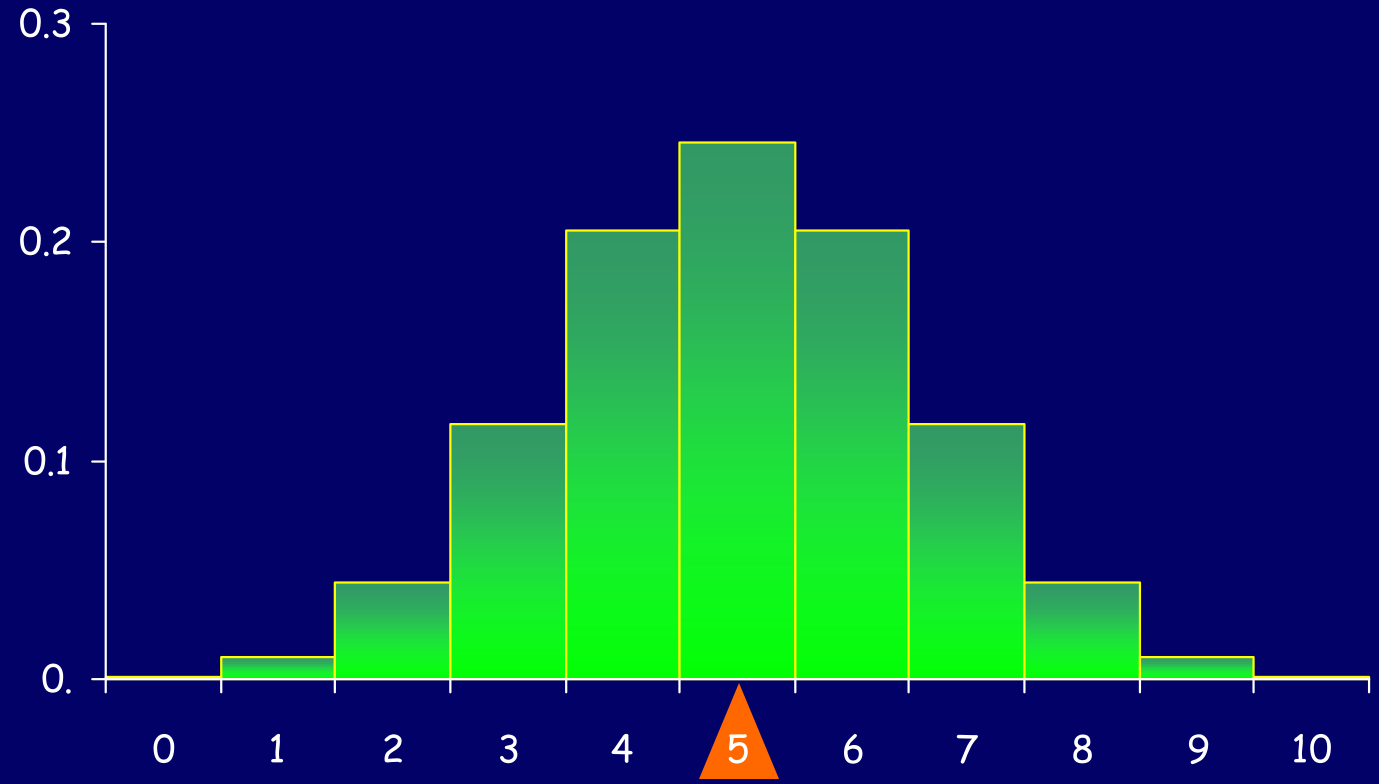
$$+6*210\left(\frac{1}{2}\right)^{10} + 7*120\left(\frac{1}{2}\right)^{10} + 8*45\left(\frac{1}{2}\right)^{10} + 9*10\left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10}(10+90+360+840+1260+1260+840+360+90+10)$$

$$= \frac{5120}{1024} = 5$$

Binomial(10, ½)

We could also look at the histogram:



Addition Rule

- For any two random variables X and Y defined over the same sample space

$$E(X+Y) = E(X) + E(Y).$$

- Consequently, for a sequence of random variables $X_1, X_2, \ldots, X_n$,

$$E(X_1+X_2+\ldots+X_n) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

- Therefore the mean of Bin(10,1/2) = 5.

Multiplication Rule and Inequalities

- Multiplication rule: $E[aX] = a\,E[X]$.
- $E[aX] = \sum_x a\,x\,P[X = x] = a \sum_x P[X = x] = a\,E[x]$

- If $X \geq Y$ then $E[X] \geq E[Y]$.
- This follows since X-Y is non negative and
- $E[X] - E[Y] = E[X-Y] \geq 0$.

Sum of Two Dice

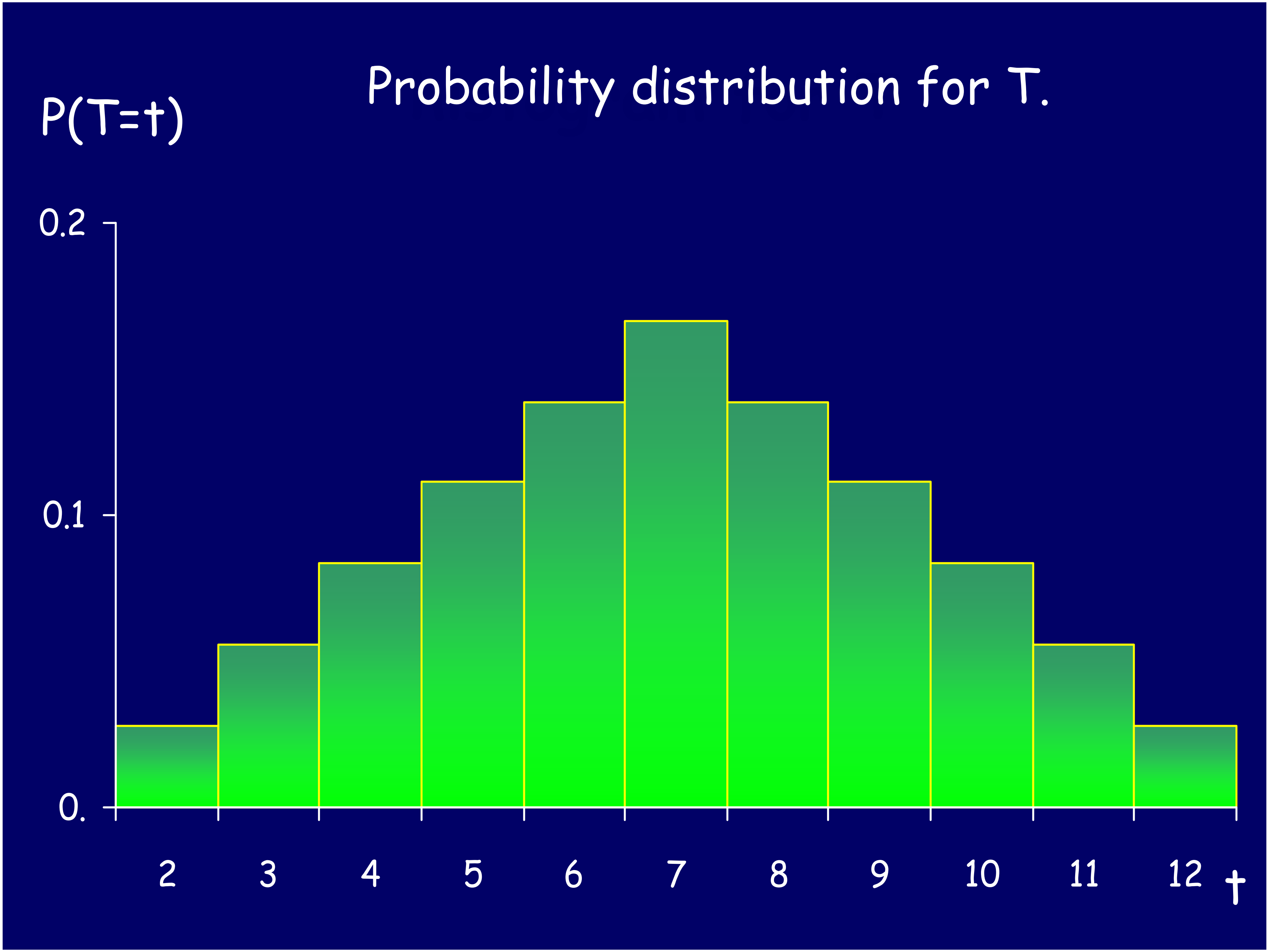
- Let T be the sum of two dice. What's E(T)?

- The "easy" way:

$$E(T) = \sum_t tP(T=t).$$

This sum will have 11 terms.

- We could also find the center of mass of the histogram (easy to do by symmetry).


Probability distribution for T.
P(T=t)
0.2
0.1
0.
2
3
4
5
6
7
8
9
10
11
12
t

Sum of Two Dice

- Or we can using the addition rule:

$T=X_1 + X_2$, where X_1 = 1st role, X_2 = 2nd:

$$E(T) = E(X_1)+ E(X_2) = 3.5+3.5 = 7.$$

Indicators

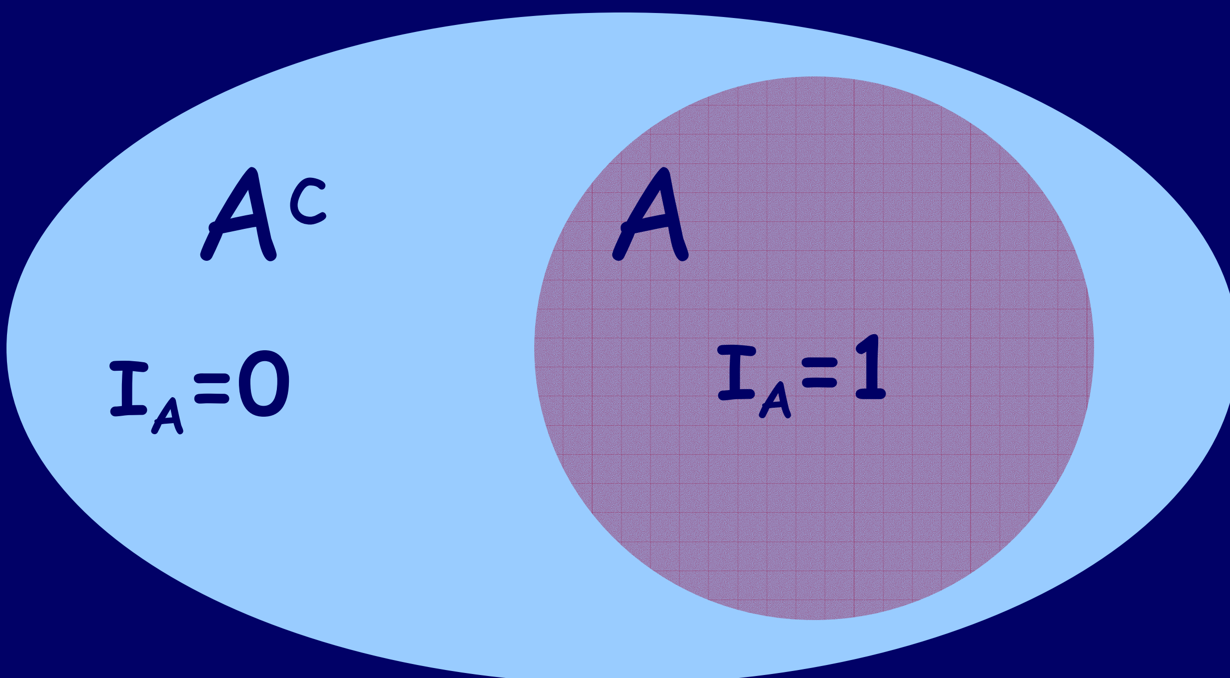
Indicators associate 0/1 valued random variables to events.

Definition: The indicator of the event A,

I_A is the random variable that takes the value 1 for outcomes in A and the value 0 for outcomes in A^c.

Indicators

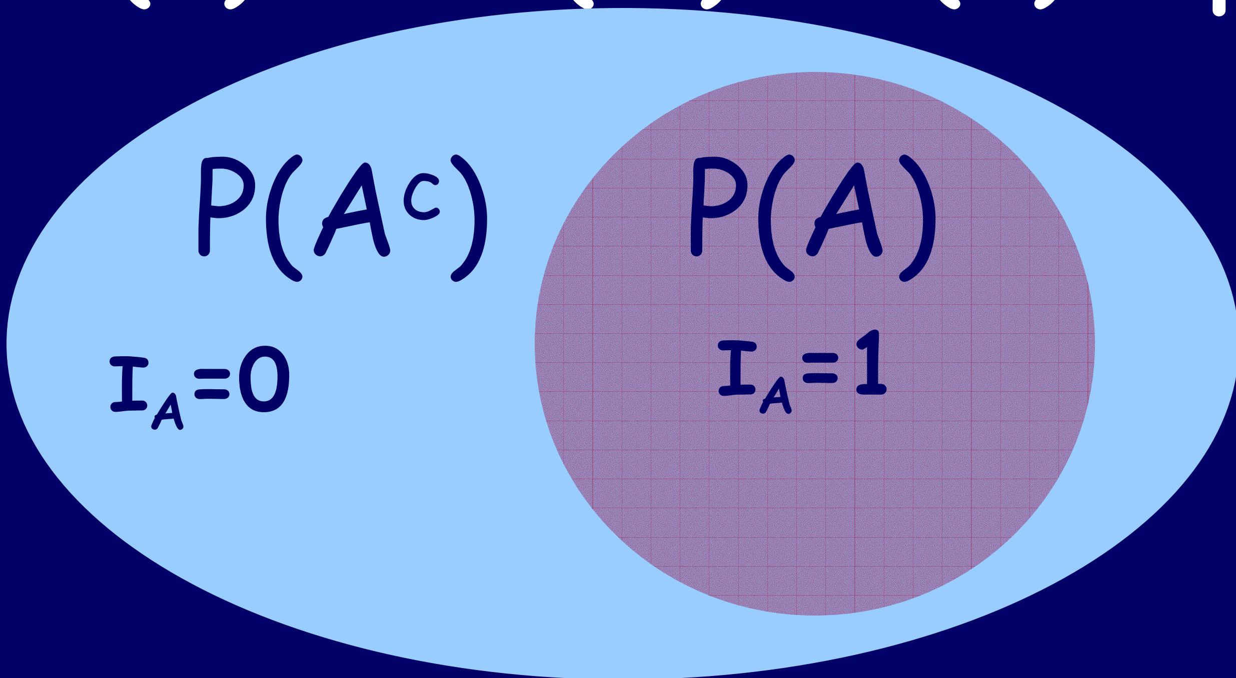
Suppose I_A is an indicator of an event A with probability p.



Expectation of Indicators

Then:

$E(I_A) = 1*P(A) + 0*P(A^c) = P(A) = p.$



Expected Number of Events that Occur

- Suppose there are n events $A_1, A_2, \ldots, A_n$.
- Let $X = I_1 + I_2 + \ldots + I_n$ where I_i is the indicator of A_i
- Then X counts the number of events that occur.
- By the addition rule:

$$E(X) = P(A_1) + P(A_2) + \ldots P(A_n).$$

Repeated Trials

- Let X^i = indicator of success on the i^{th} coin toss ($X^i = 1$ if the i^{th} coin toss = H head, and $X^i = 0$ otherwise).
- The sequence $X^1, X^2, \cdots, X^n$ is a sequence of n independent variables with Bernoulli(p) distribution over {0,1}.
- The number of heads in n coin tosses given by $S_n = X^1 + X^2 + \cdots + X^n$.
- $E(S_n) = nE(X^i) = np$
- Thus the mean of Bin(n,p) RV = np.

Expected Number of Aces

- Let Y be the number of aces in a poker hand.
- Then:

$Y = I_{1st\ ace} + I_{3rd\ ace} + I_{4th\ ace} + I_{5th\ ace} + I_{2nd\ ace}$.

- And: E(Y) = 5*P(ace) = 5*4/52 = 0.385.
- Alternatively, since Y has the hyper-geometric distribution we can calculate:

$$E(Y) = \sum_{x=0}^{4} y \frac{\binom{4}{y}\binom{48}{5-y}}{\binom{52}{5}}$$

Non-negative Integer Valued RV

- Suppose X is an integer valued, non-negative random variable.
- Let $A_i = \{X \geq i\}$ for $i=1,2,\ldots;$
- Let I_i the indicator of the set A_i.
- Then

$$X=\sum_i I_i.$$

Non-negative Integer Valued RV

- The equality

$$X(\text{outcome})=\sum_i I_i(\text{outcome})$$

follows since if X(outcome) = i, then

outcome $\in A_1 \cap A_2 \cap \ldots \cap A_{i.}$ but not to A_j, j>i.

- So $(I_1+I_2+\ldots+I_i+I_{i+1}+\ldots)(\text{outcome})$ =

$$1+1+\ldots+1+0+0+\ldots = i.$$

Tail Formula for Expectation

Let X be a non-negative integer valued RV,

Then:

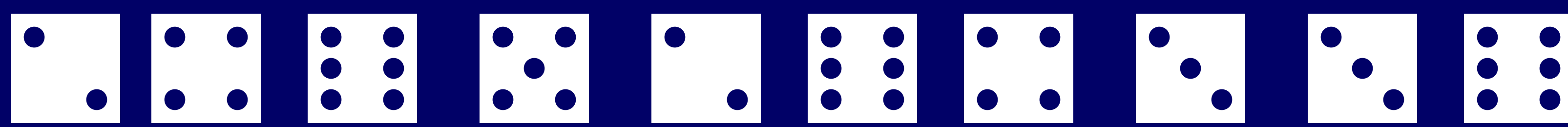
$$E(X) = E\left(\sum_i I_i\right) = \sum_i E(I_i)$$

$$= \sum_i P(X \geq i), \quad i=1,2,3\ldots$$

…					…
$P(X\geq 4)$				$P(X=4)$	…
$P(X\geq 3)$			$P(X=3)$	$P(X=4)$	…
$P(X\geq 2)$		$P(X=2)$	$P(X=3)$	$P(X=4)$	…
$P(X\geq 1)$	$P(X=1)$	$P(X=2)$	$P(X=3)$	$P(X=4)$	…
$E(X) =$	$1*P(X=1)$	$2*P(X=2)$	$3*P(X=3)$	$4*P(X=4)$	…

Minimum of 10 Dice

Suppose we roll a die 10 times and let X be the minimum of the numbers rolled.



Here X = 2.

Question: what's the expected value of X?

Minimum of 10 Dice

- Let's use the tail formula to compute E(X):

$$E(X) = \sum_i P(X \geq i).$$

$P(X \geq 1) = 1;$

$P(X \geq 2) = (5/6)^{10};$

$P(X \geq 3) = (4/6)^{10};$

$P(X \geq 4) = (3/6)^{10};$

$P(X \geq 5) = (2/6)^{10};$

$P(X \geq 6) = (1/6)^{10}$

$$E(X) = (6^{10}+5^{10}+4^{10}+3^{10}+4^{10}+3^{10})/6^{10}$$

$$= 1.17984$$

Indicators

- If the events A_1, A_2, …, A_j are mutually exclusive then

$$I_1 + I_2 + \ldots + I_j = I_{A_1 \cup A_2 \cup \ldots \cup A_j}$$

- And

$$P(\cup^j_{i=1} A_i) = \sum_i P(A_i).$$

Tail Formula for Expectation

Let X be a non-negative integer valued RV,

Then:

$$E(X) = E\left(\sum_i I_i\right) = \sum_i E(I_i)$$

$$= \sum_i P(X \geq i), \quad i=1,2,3\ldots$$

Boole's Inequality

For a non-negative integer valued X we can obtain Boole's inequality:

$$P(X \geq 1) \leq \sum_i P(X \geq i) = E(X)$$

Markov's Inequality

Markov inequality:

If $X \geq 0$, then for every $a > 0$

$$P(X \geq a) \leq E(X)/a.$$

- This is proven as follows.
- Note that if $X \geq Y$ then $E(X) \geq E(Y)$.
- Take Y = indicator of the event $\{X \geq a\}$.
- Then $E(Y) = P(X \geq a)$ and $X \geq aY$ so:
- $E(X) \geq E(aY) = a\,E(Y) = a\,P(X \geq a)$.

Expectation of a Function of a Random Variable

- For any function g defined on the range space of a random variable X with a finite number of values

$$E[g(X)] = \sum_x g(x)\, P(X=x).$$

Proof:

- Note that:

$$P(g(X)=y)= \sum_{\{x:g(x)=y\}} P(X=x).$$

- Therefore:

$$E[g(X)] = \sum_y y\, P(g(X)=y) = \sum_y \sum_{\{x:g(x)=y\}} g(x)P(X=x)$$
$$= \sum_x g(x)\, P(X=x).$$

Expectation of a Function of a Random Variable

- Constants:

$$g(X)=c \Rightarrow E[g(x)]=c.$$

- Linear functions:

$$g(X)=aX + b \Rightarrow E[g(x)]=aE(X)+b.$$

(These are the only cases when $E(g(X)) = g(E(X))$.)

Expectation of a Function of a Random Variable

- Monomials:

$$g(X)=X^k \Rightarrow E[g(x)]=\sum_x x^k P(X=x).$$

$\sum_x x^k P(X=x)$ is called the k^{th} **moment** of X.

Expectation of a Function of a Random Variable

Question: For X representing the number on a die, what is the second moment of X?

$\sum_x x^2 P(X=x) = \sum_x x^2/6 = 1/6*(1 + 4 + 9 + 16 + 25 + 36)$

$= 91/6 = 15.16667$

Expectation of a Function of Several Random Variables

- If X and Y are two random variables we obtain:

$$E(g(X,Y)) = \sum_{\{\text{all } (x,y)\}} g(x,y)P(X=x, Y=y).$$

This allows to prove that E[X+Y] = E[X] + E[Y]:

$$E(X) = \sum_{\{\text{all } (x,y)\}} x\, P(X=x, Y=y);$$

$$E(Y) = \sum_{\{\text{all } (x,y)\}} y\, P(X=x, Y=y);$$

$$E(X+Y) = \sum_{\{\text{all } (x,y)\}} (x+y)\, P(X=x, Y=y);$$

$$E(X+Y) = E(X) + E(Y)$$

Expectation of a Function of Several Random Variables

$$E(g(X,Y)) = \sum_{\{all\ (x,y)\}} g(x,y)P(X=x,\ Y=y).$$

Product:

$$E(XY) = \sum_{\{all\ (x,y)\}} xy\ P(X=x,\ Y=y);$$

$$E(XY) = \sum_x \sum_y xy\ P(X=x,\ Y=y);$$

Is $E(XY) = E(X)E(Y)$?

Product Rule for Independent Random Variables

- However, if X and Y are independent,

$$P(X=x,Y=y)=P(X=x)P(Y=y)$$

then product formula simplifies:

$$E(XY) = \sum_x \sum_y xy\, P(X=x)\, P(Y=y)$$

$$= \left(\sum_x x\, P(X=x)\right) \left(\sum_y y\, P(Y=y)\right) =$$

$$E(X)\, E(Y)$$

- If X and Y are independent then:

$$E(XY) = E(X)\, E(Y);$$

Expectation interpretation as a Long-Run Average

- If we repeatedly sample from the distribution of X then $P(X=x)$ will be close to the observed frequency of x in the sample.

- $E(X)$ will be approximately the long-run average of the sample.

Mean, Mode and Median

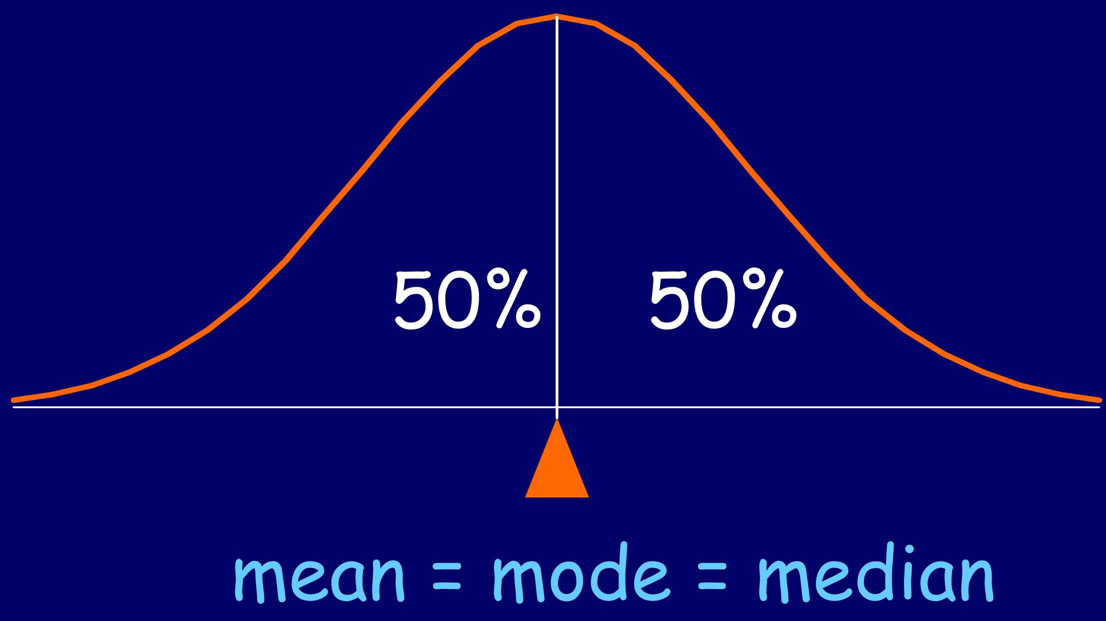
- The Mode of X is the most likely possible value of X.
- The mode need not be unique.

- The Median of X is a number m such that both $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$.
- The median may also not be unique.

- Mean and Median are not necessarily possible values (mode is).

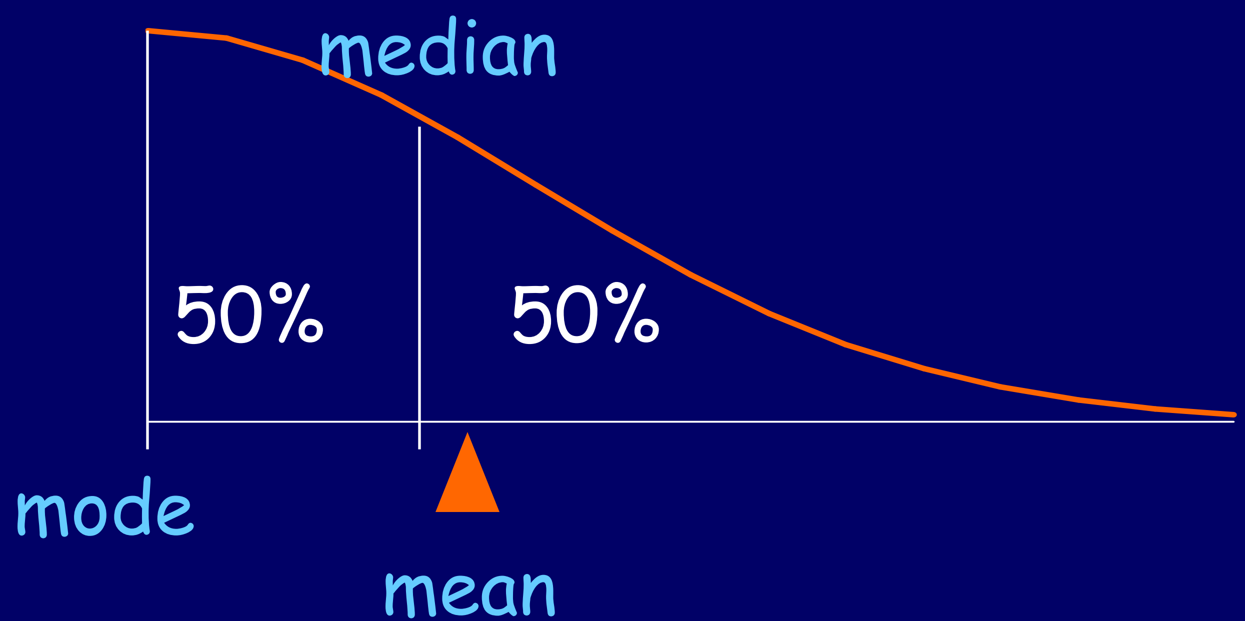
Mean, Mode and Median

For a symmetrical distribution, which has a unique Mode, all three: Mean, Mode and Median are the same.

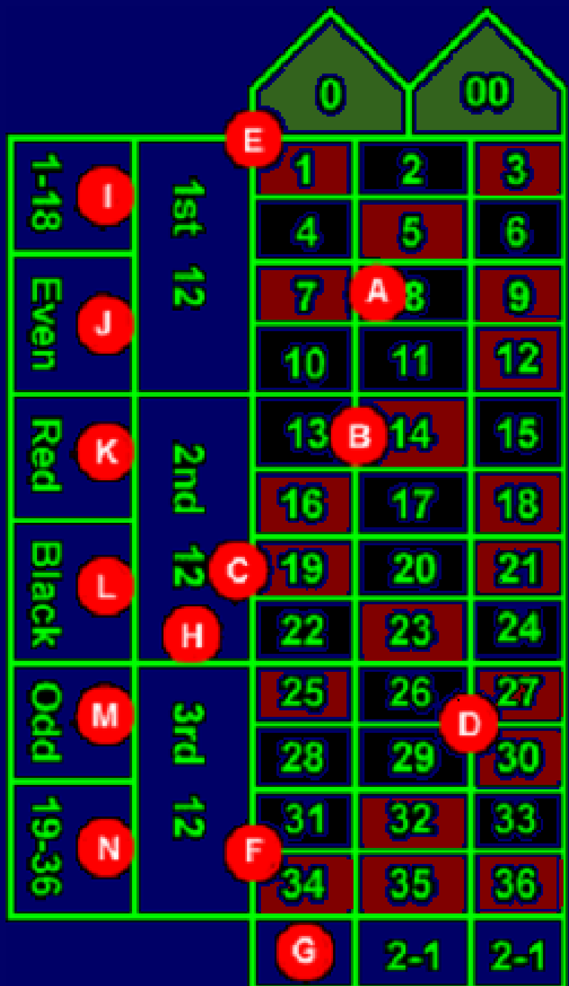


Mean, Mode and Median

For a distribution with a long right tail Mean is greater than the Median.



Roulette



	Bet	Pay-off
•	Straight Up (one number)	35:1
•	Split	17:1
•	Line/Street (three numbers)	11:1
•	Corner (four numbers)	8:1
•	Five line (5 numbers-0,00,1,2 or 3)	6:1
•	Six line (six numbers)	5:1
•	Column (twelve numbers)	2:1
•	Dozens (twelve numbers)	2:1
•	1 - 18	1:1
•	Even	1:1
•	Red	1:1
•	Black	1:1
•	Odd	1:1
•	19-36	1:1

Betting on Red

Suppose we want to be $1 on Red. Our chance of winning is 18/38.

Question:

What should be the pay-off to make it a fair bet?

Betting on Red

This question really only makes sense if we repeatedly bet \$1 on Red.

Suppose that we could win \$x if Reds come up and lose \$1, otherwise. If X denotes our returns then $P(X=x) = 18/38$; $P(X=-1)=20/38$.

In a fair game, we expect to break even on average.

Betting on Red

Our expected return is:

$$x*18/38 - 1*20/38.$$

Setting this to zero gives us

$$x=20/18=1.1111111\ldots .$$

This is greater than the pay-off of 1:1 that is offered by the casino.