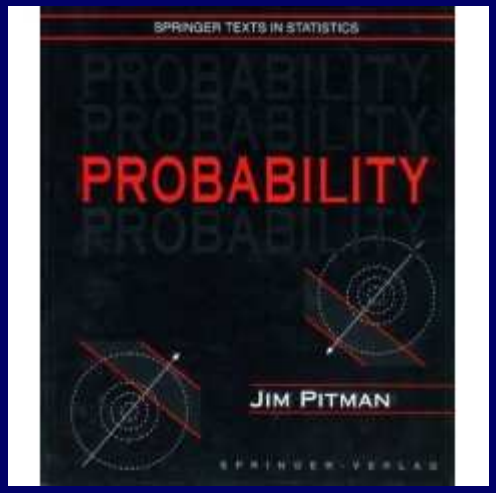
Introduction to probability

Stat 134 FAll 2005

Berkeley



Follows Jim Pitman's book:

Probability

Section 2.5

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Sampling with replacement

- Suppose we have a population of size N with G good elements and B bad elements. We draw n times *with* replacement from this population.
- The number g of good elements in the sample will have a binomial(n,p) distribution with p=G/N and 1-p = B/N

$$P(g \text{ good and } b \text{ bad}) = \binom{n}{g} \frac{(G)^g (B)^b}{(N)^n}.$$

Sampling with replacement

- If n is large, this will be well approximated by N(np, $\sqrt{np(1-p)}$).
- The proportion of good elements in the sample g/n will lie in the interval $p \pm \frac{1}{\sqrt{n}}$ with probability 95%.
- If the p is not known, it can be estimated by the method of confidence intervals.

Confidence intervals

Suppose we observe the results of n trials with an unknown probability of success p.

The observed frequency of successes $\hat{p} = \frac{\#\text{successes}}{n}$.

The Normal Curve Approximation says that for any fixed p and n large enough, there is a 99.99% chance that the observed frequency $\hat{p}$ will differ from p by less than $4\sqrt{\frac{p(1-p)}{n}}$.

It's easy to see that $\sqrt{p(1-p)} \leq \frac{1}{2}$, so $4\sqrt{\frac{p(1-p)}{n}} \leq \frac{2}{\sqrt{n}}$.

$$\left(\hat{p} - \frac{2}{\sqrt{n}}, \hat{p} - \frac{2}{\sqrt{n}}\right)$$

is called a **99.99% confidence interval.**

Sampling without replacement

- Let's now think about drawing *without* replacement. The sample size has to be restricted to $n \leq N$.

- Then number of possible orderings of n elements out of N is:

$$(N)_n = N(N-1)(N-2) \ldots (N-n+1).$$

- $(N)_n$ is called N order n

Sampling without replacement

Note that:

$$\binom{N}{n} = \frac{(N)_n}{n!}.$$

So:

$$(N)_n = \binom{N}{n} n!$$

Sampling without replacement

- The chance of getting a specific sample with g good elements followed by b bad ones is:

$$\frac{G}{N}\cdot\frac{G-1}{N-1}\cdots\frac{G-g+1}{N-g+1}\cdot\frac{B}{N-g}\cdot\frac{B-1}{N-g-1}\cdots\frac{B-b+1}{N-g-b+1}=\frac{(G)_g(B)_b}{(N)_n}.$$

- Since there are $\binom{n}{g}$ samples with g good and b bad elements all having the same probability, we obtain:

Sampling with and without replacement

- For sampling without replacement:

$$P(g \text{ good and } b \text{ bad}) = \binom{n}{g} \frac{(G)_g (B)_b}{(N)_n} = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}}.$$

- For sampling with replacement:

$$P(g \text{ good and } b \text{ bad}) = \binom{n}{g} \frac{(G)^g (B)^b}{(N)^n}.$$

Hypergeometric Distribution.

- The distribution of the number of good elements in a sample of
  - size n
  - without replacement
- From a population of
  - G good and
  - N-G = B bad elements

Is called the

hypergeometric distribution with parameters (n,N,G).

Sampling with and without replacement

- When N is large $(N)_n$ / $N^n \to 1$.
- When B is large $(B)_b$ / $B^b \to 1$
- When G is large $(G)_g$ / $G^g \to 1$

So for fixed b,g and n as B,G,N $\to \infty$ the hypergeometric distribution can be approximated by a binomial(n,G/N).

Multinomial Distribution

Suppose each trial can result in m possible categories $c_1, c_2, \ldots, c_m$ with probabilities $p_1, p_2, \ldots, p_m$, where $p_1+p_2+\ldots+p_m = 1$.

Suppose we make a sequence of n independent trials and let N_i denote the number of results in the i^{th} category c_i.

Multinomial Distribution

Then for every m-tuple of non-negative integers $(n_1, n_2, \ldots, n_m)$ with $n_1+n_2+\ldots+n_m = n$

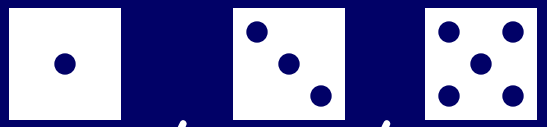
$$P(N_1=n_1, N_2=n_2, \ldots, N_m=n_m) = \frac{n!}{n_1!n_2!\ldots n_m!} p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}$$

Probability of any specific sequence

Number of possible sequences with the given elements

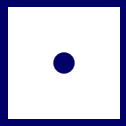
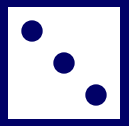
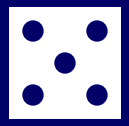
1,3,5 and 'even'

Suppose we roll a fair die 10 times and record the number of

 and even's.

Question:

What's the probability of seeing

1  , 2  , 3  and 4 even numbers?

1,3,5 and 'even'

Using the multinomial distribution:

$$P(N_1=1,N_3=2,N_5=3,N_{even}=4) = \frac{10!}{1!2!3!4!}\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^3\left(\frac{3}{6}\right)^4$$

$$= 0.016878858$$

Hypergeometric Extension

- Consider also: this is the no. of ways to ***choose*** (g things from G) and (b bad things from B), out of all possible ways to ***choose*** n things from N.

- This way of counting lets us generalize to multiple subcategories easily. How many ways are there to choose g good from G and b bad from B and o ok's from O and p passable from P out of all possible ways to choose n things from N?