Introduction to probability

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Follows Jim Pitman's book:
Probability
Sections 1.6
Multiplication rule for 3 Events

The Multiplication rule for two events says:
\[ P(AB) = P(A) P(B \mid A) \]

The Multiplication rule extends to 3 Events:
\[ P(ABC) = P(AB)P(C \mid AB) = P(A) P(B \mid A) P(C \mid AB) \]
Similarly, it extends to n events:

\[ P(A_1 A_2 \ldots A_n) = P(A_1 \ldots A_{n-1})P(A_n|A_1 \ldots A_{n-1}) \]

\[ = P(A_1) P(A_2|A_1) P(A_3|A_1 A_2) \ldots P(A_n|A_1 \ldots A_{n-1}) \]
Shesh Besh Backgammon

We roll two dice. What is the chance that we will roll out Shesh Besh: 🟢🟠 for the *first* time on the *n*’th roll?

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<th>N=1</th>
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<th>3</th>
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P = \[
\frac{1}{36} \quad \frac{3 \ 5 \ 1}{36 \ 36 \ 36} \quad \frac{3 \ 5 \ 3 \ 5 \ 1}{36 \ 36 \ 36 \ 36} \quad \frac{3 \ 5 \ 3 \ 5 \ 3 \ 5 \ 1}{36 \ 36 \ 36 \ 36 \ 36}
\]
This is called a Geometric Distribution with parameter $p=1/36$. 

The graph illustrates the probability distribution for the Geometric Distribution with parameter $p=1/36$. The horizontal axis represents the number of trials until the first success, while the vertical axis shows the probability of each outcome. The distribution is plotted from 0 to 100 trials, and the probabilities decrease as the number of trials increases.
The Geometric distribution

In $\text{Geom}(p)$ distribution the probability of the outcome $n$ for $n=1,2,3...$ is given by:

$$p \cdot (1-p)^{n-1}$$

Sanity check: is $\sum_{n=1}^{\infty} p(1-p)^{n-1} = 1$?

This is used often to model the wait time in situations where we are waiting for something (that has a certain chance of happening) to happen.
The Birthday Problem

If there are $n$ students in the class, what is the chance that at least two of them have the same birthday?

$$P(\text{at least 2 have same birthday}) = 1 - P(\text{No coinciding birthdays}).$$

Let $B_i$ be the birthday of student number $i$.

The probability of no coinciding birthdays is:

$$P(B_2 \notin \{B_1\} \& B_3 \notin \{B_1, B_2\} \& \ldots \& B_n \notin \{B_1, \ldots, B_{n-1}\}).$$
Use multiplication rule to find

\[ P(B_2 \notin \{B_1\} \land B_3 \notin \{B_1,B_2\} \land \ldots \land B_n \notin \{B_1,\ldots,B_{n-1}\}) \].
The Birthday Problem

P(at least 2 have same birthday) =
1 - P(No coinciding birthdays) =

\[1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\ldots\left(1 - \frac{n-1}{365}\right)\]

Q: How can we compute this for large n?
A: Approximate!
The Birthday Problem

\[
\log(P(\text{No coinciding birthdays})) =
\]

\[
= \log((1 - \frac{1}{365})(1 - \frac{2}{365})\ldots(1 - \frac{n-1}{365}))
\]

\[
= \log(1 - \frac{1}{365}) + \log(1 - \frac{2}{365}) + \ldots + \log(1 - \frac{n-1}{365})
\]

\[
\approx - \frac{1}{365} - \frac{2}{365} - \ldots - \frac{n-1}{365}
\]

\[
= - \frac{1}{365} \left( \frac{1}{2} n(n - 1) \right)
\]
The Birthday Problem

\[ P(\text{No coinciding birthdays}) \approx e^{\frac{-n(n-1)}{2 \times 365}} \]

\[ P(\text{At least 2 have same birthday}) \approx 1 - e^{\frac{-n(n-1)}{2 \times 365}} \]
Probability of no coinciding birthday as a function of $n$
Independence of 3 events

Recall that A and B are independent if:
\[ P(B|A) = P(B|A^c) = P(B); \]

We say that A, B and C are independent if:
\[ P(C|AB) = P(C|A^cB) = P(C|A^cB^c) = P(C|AB^c) = P(C) \]
Independence of n events

The events $A_1, \ldots, A_n$ are independent if

$$P(A_i \mid B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n) = P(A_i)$$

for $B_i = A_i$ or $A_i^c$

This is equivalent to following multiplication rules:

$$P(B_1 B_2 \ldots B_n) = P(B_1) P(B_2) \ldots P(B_n)$$

for $B_i = A_i$ or $A_i^c$
Independence of $n$ events

**Question**: Consider the events $A_1, \ldots, A_n$.
Suppose that for all $i$ and $j$ the events $A_i$ and $A_j$ are independent.

Does that mean that $A_1, \ldots, A_n$ are all independent?
Pair-wise independence does not imply independence

I pick one of these people at random. If I tell you that it’s a girl, there is an equal chance that she is a blond or a brunet; she has blue or brown eyes. Similarly for a boy.

However, if I tell you that I picked a blond and blue-eyed person, it has to be a boy. So sex, eye color and hair color, for this group, are pair-wise independent, but not independent.