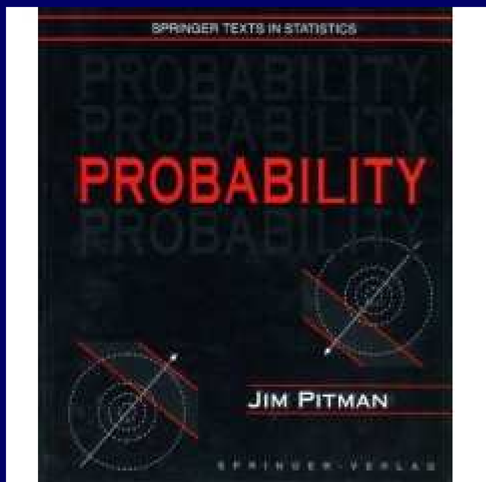
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Sections 1.6

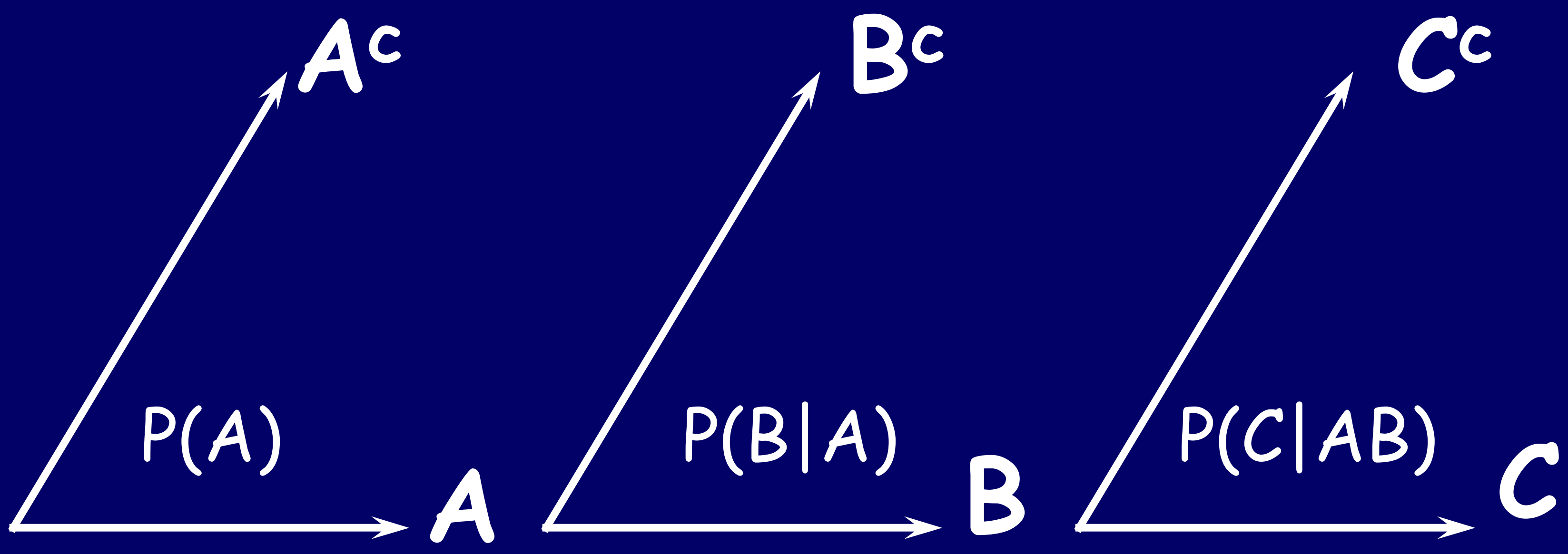
Multiplication rule for 3 Events

The Multiplication rule for two events says:

$$P(AB) = P(A)\,P(B \mid A)$$

The Multiplication rule extends to 3 Events:

$$P(ABC) = P(AB)P(C \mid AB) = P(A)\,P(B \mid A)\,P(C \mid AB)$$

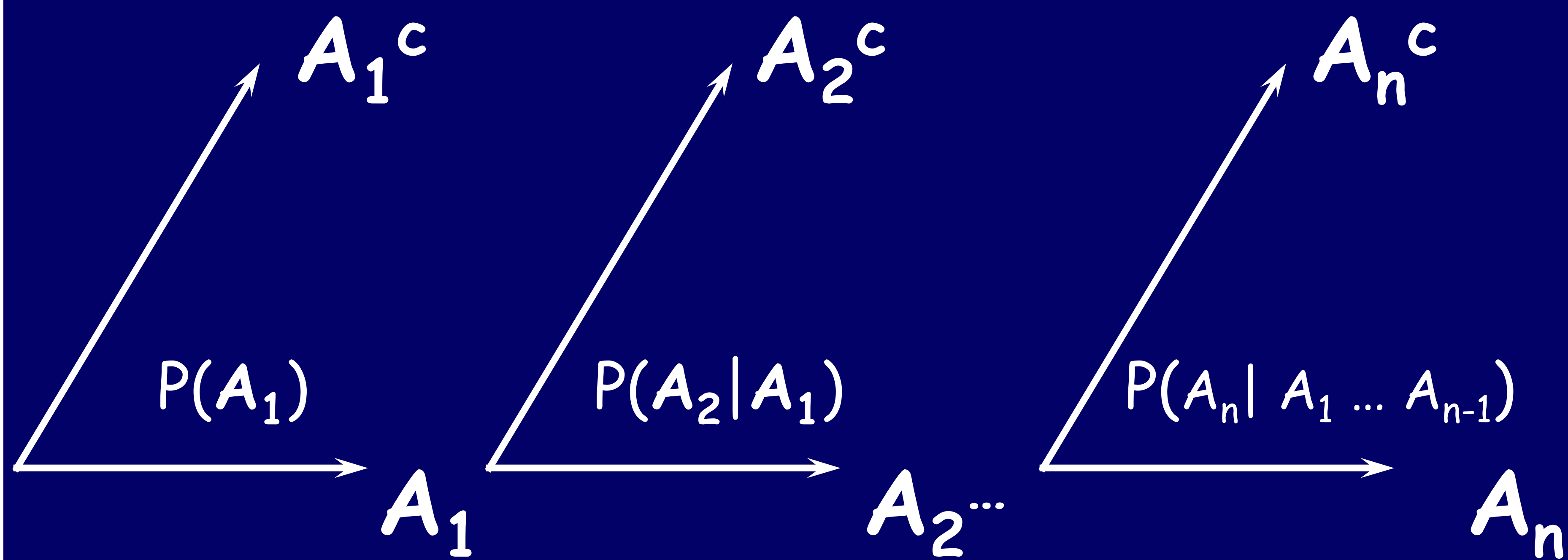


Multiplication rule for n Events

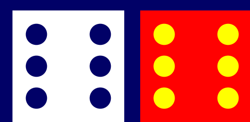
Similarly, it extends to n events:

$$P(A_1 A_2 \ldots A_n) = P(A_1 \ldots A_{n-1})P(A_n | A_1 \ldots A_{n-1})$$

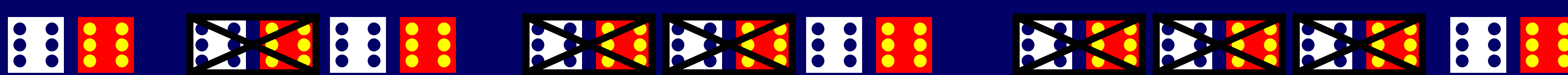
$$= P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 A_2) \ldots P(A_n| A_1 \ldots A_{n-1})$$



Shesh Besh Backgammon

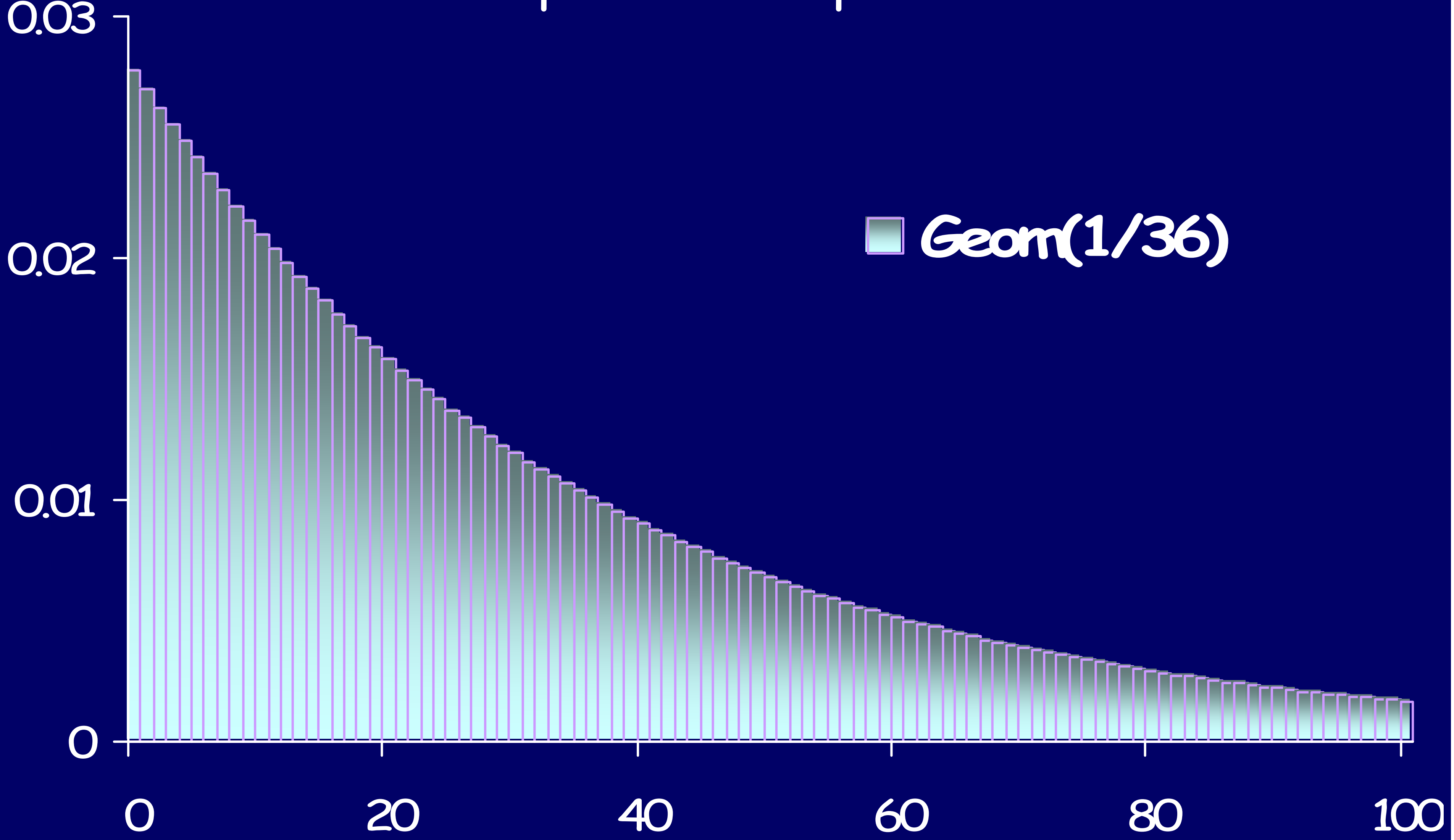
We roll two dice. What is the chance that we will roll out Shesh Besh:  for the ***first*** time on the n'th roll?

N=1 2 3 4



$$P = \frac{1}{36} \qquad \frac{35}{36}\,\frac{1}{36} \qquad \frac{35}{36}\,\frac{35}{36}\,\frac{1}{36} \qquad \frac{35}{36}\,\frac{35}{36}\,\frac{35}{36}\,\frac{1}{36}$$

This is called a Geometric Distribution with parameter p=1/36.



The Geometric distribution

In Geom(p) distribution the probability of the outcome n for

n=1,2,3… is given by:

$$p\,(1-p)^{n-1}$$

Sanity check: is $\sum_{n=1}^{\infty} p(1-p)^{n-1} = 1$?

This is used often to model the wait time in situations where we are waiting for something (that has a certain chance of happening) to happen.

The Birthday Problem

If there are n students in the class, what is the chance that at least two of them have the same birthday?

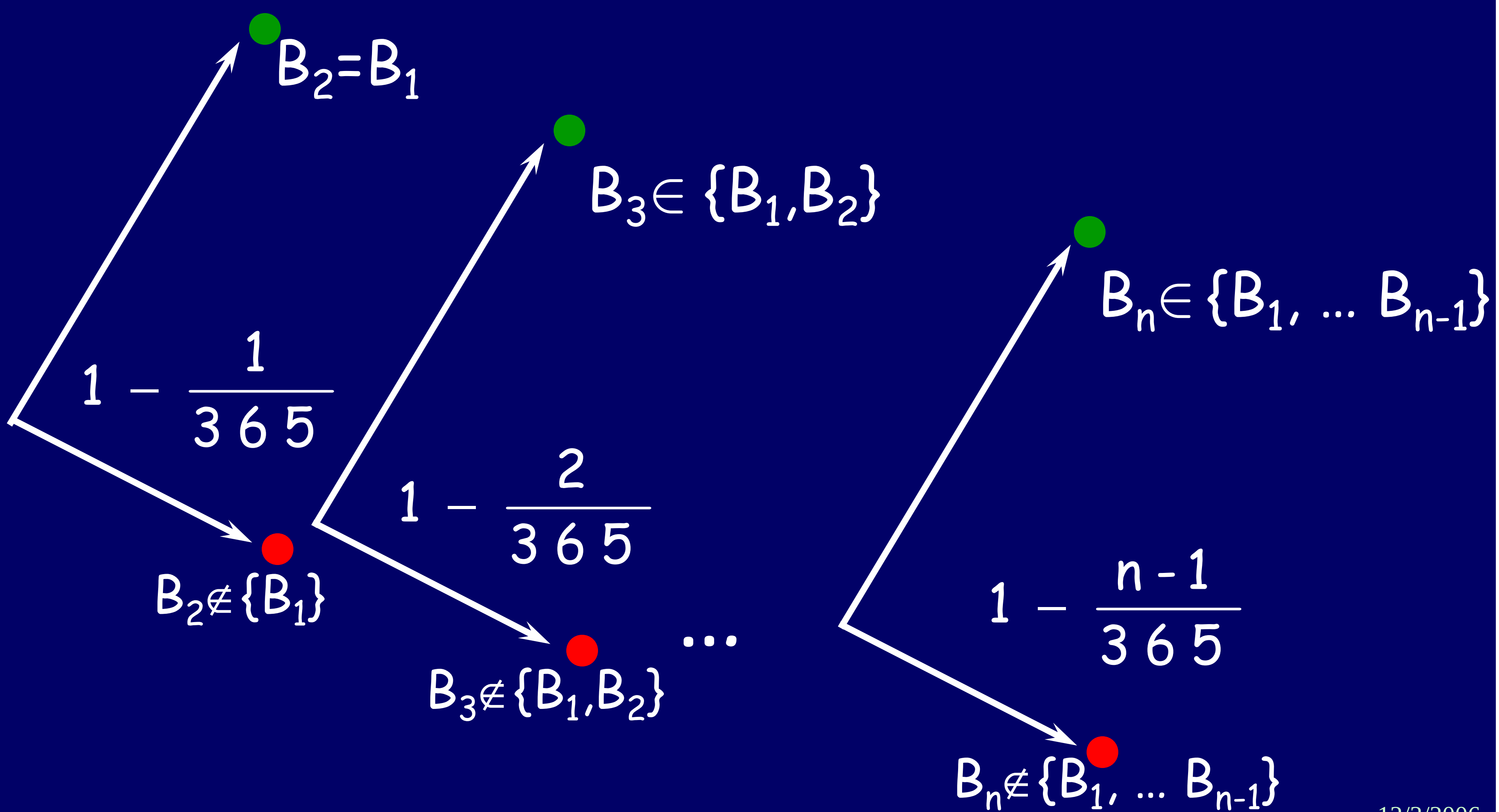
P(at least 2 have same birthday) =
1 – P(No coinciding birthdays).

Let B_i be the birthday of student number i.
The probability of no coinciding birthdays is:

$$P(B_2 \notin \{B_1\} \ \& \ B_3 \notin \{B_1,B_2\} \ \& \ \ldots \ \& \ B_n \notin \{B_1,\ldots,B_{n-1}\}).$$

Use multiplication rule to find

$P(B_2 \notin \{B_1\} \;\&\; B_3 \notin \{B_1,B_2\} \;\&\; \ldots \;\&\; B_n \notin \{B_1,\ldots,B_{n-1}\})$.



The Birthday Problem

P(at least 2 have same birthday) =

1 – P(No coinciding birthdays) =

$$1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\ldots\left(1 - \frac{n-1}{365}\right)$$

Q: How can we compute this for large n?

A: Approximate!

The Birthday Problem

$\log$(P(No coinciding birthdays))=

$$= \log((1 - \frac{1}{365})(1 - \frac{2}{365})\ldots(1 - \frac{n\text{-}1}{365}))$$

$$= \log(1 - \frac{1}{365}) + \log(1 - \frac{2}{365}) + \ldots + \log(1 - \frac{n\text{-}1}{365})$$

$$\approx -\frac{1}{365} - \frac{2}{365} - \ldots - \frac{n\text{-}1}{365}$$

$$= -\frac{1}{365}(\frac{1}{2}n(n-1))$$

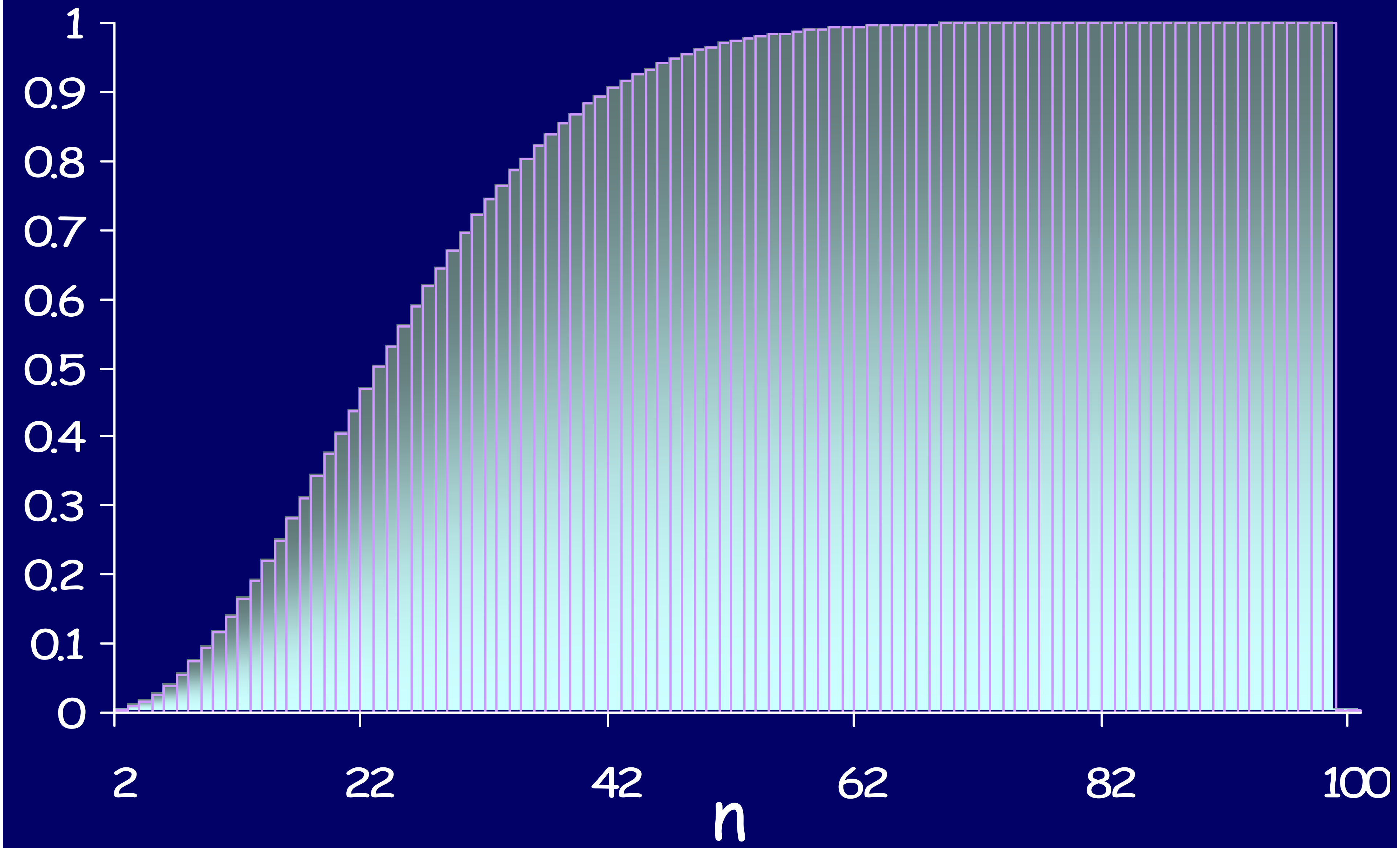
The Birthday Problem

$$P(\text{No coinciding birthdays}) \approx e^{-\frac{n(n-1)}{2\times 365}}$$

$$P(\text{At least 2 have same birthday})$$

$$\approx 1 - e^{-\frac{n(n-1)}{2\times 365}}$$

Probability of no coinciding birthday as a function of n



Independence of 3 events

Recall that A and B are independent if:

$$P(B|A)=P(B|A^c) = P(B);$$

We say that <u>A, B and C are independent</u> if:

$$P(C|AB)= P(C|A^cB) = P(C|A^cB^c) = P(C|AB^c) =P(C)$$

Independence of n events

The events $\underline{A_1,\ldots,A_n \text{ are independent}}$ if

$$P(A_i \mid B_1,\ldots B_{i-1},B_{i+1},\ldots B_n)= P(A_i)$$

$$\text{for } B_i = A_i \text{ or } A_i^c$$

This is equivalent to following multiplication rules:

$$P(B_1\, B_2 \ldots B_n) = P(B_1)\, P(B_2) \ldots P(B_n)$$

$$\text{for } B_i = A_i \text{ or } A_i^c$$

Independence of n events

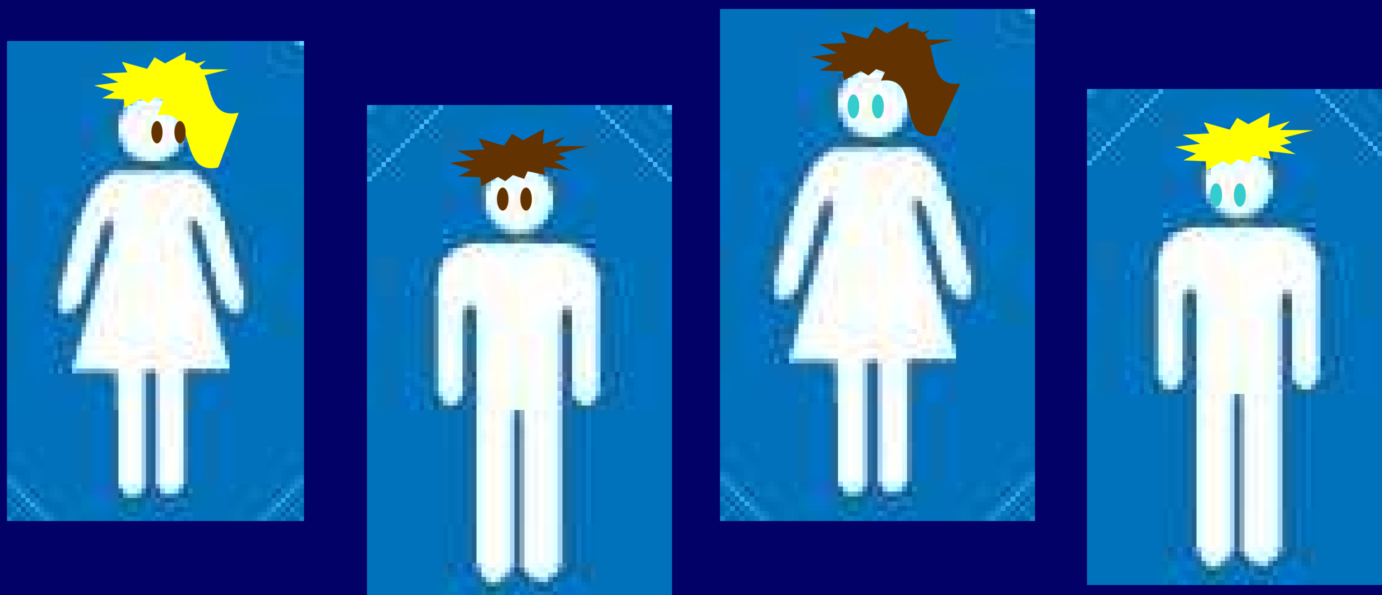
Question: Consider the events $A_1,\ldots,A_n$.

Suppose that for all i and j the events A_i and A_j are independent.

Does that mean that $A_1,\ldots,A_n$ are all independent?

Pair-wise independence does not imply independence

I pick one of these people at random. If I tell you that it's a girl, there is an equal chance that she is a blond or a brunet; she has blue or brown eyes. Similarly for a boy.



However, if a tell you that I picked a blond and blue eyed person, it has to be a boy.
So sex, eye color and hair color, for this group, are pair-wise independent, but not independent.