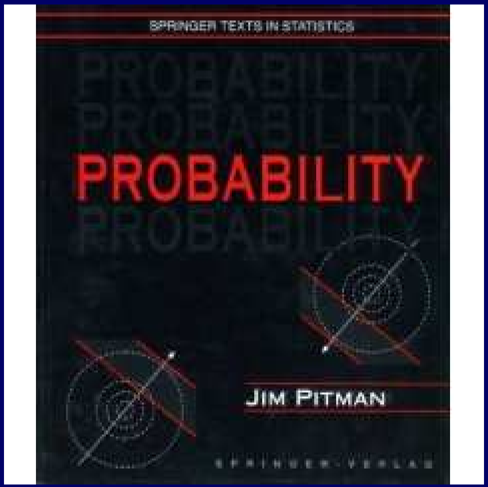
Introduction to probability

Stat 134 FAll 2006

Berkeley



Lectures prepared by:
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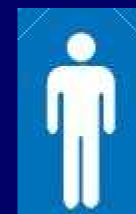
Follows Jim Pitman's book:

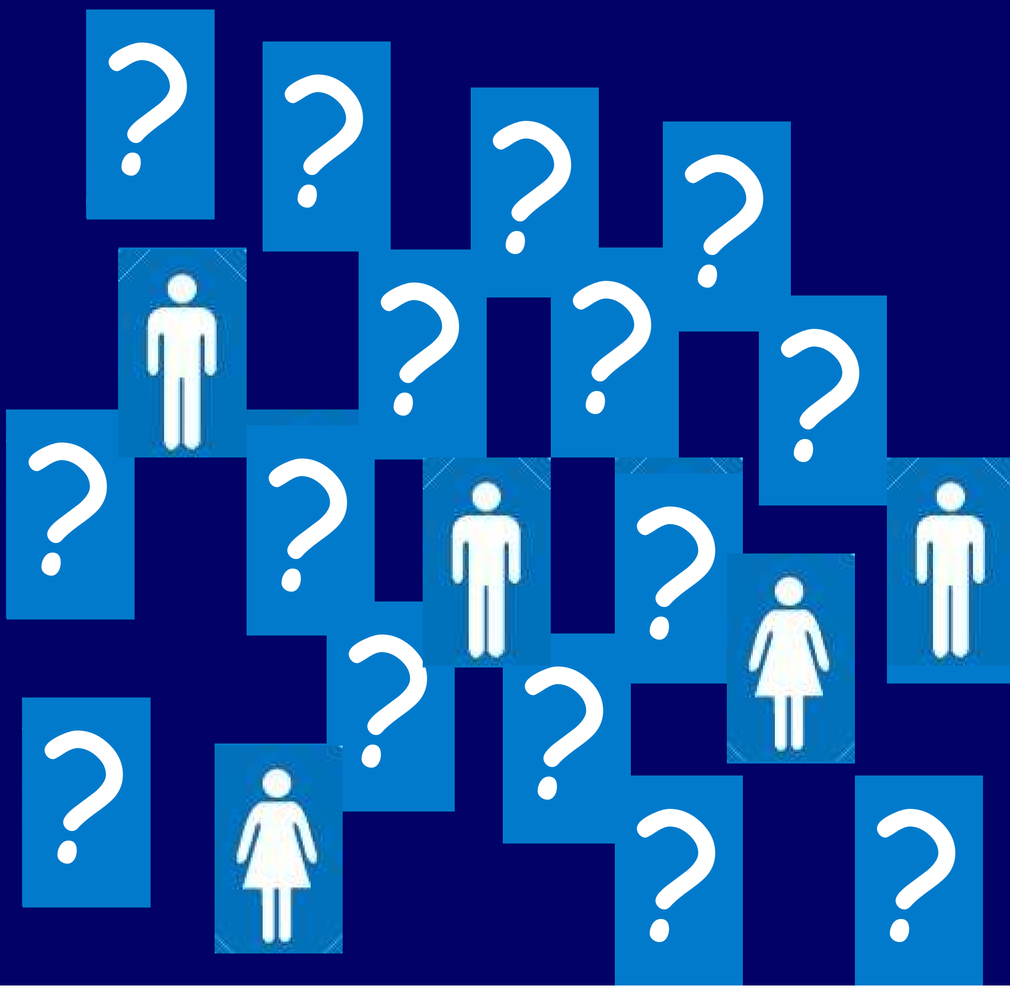
Probability

Sections 1.1-1.3

Probability as Proportion

Suppose there are 20 people taking Stat 134.

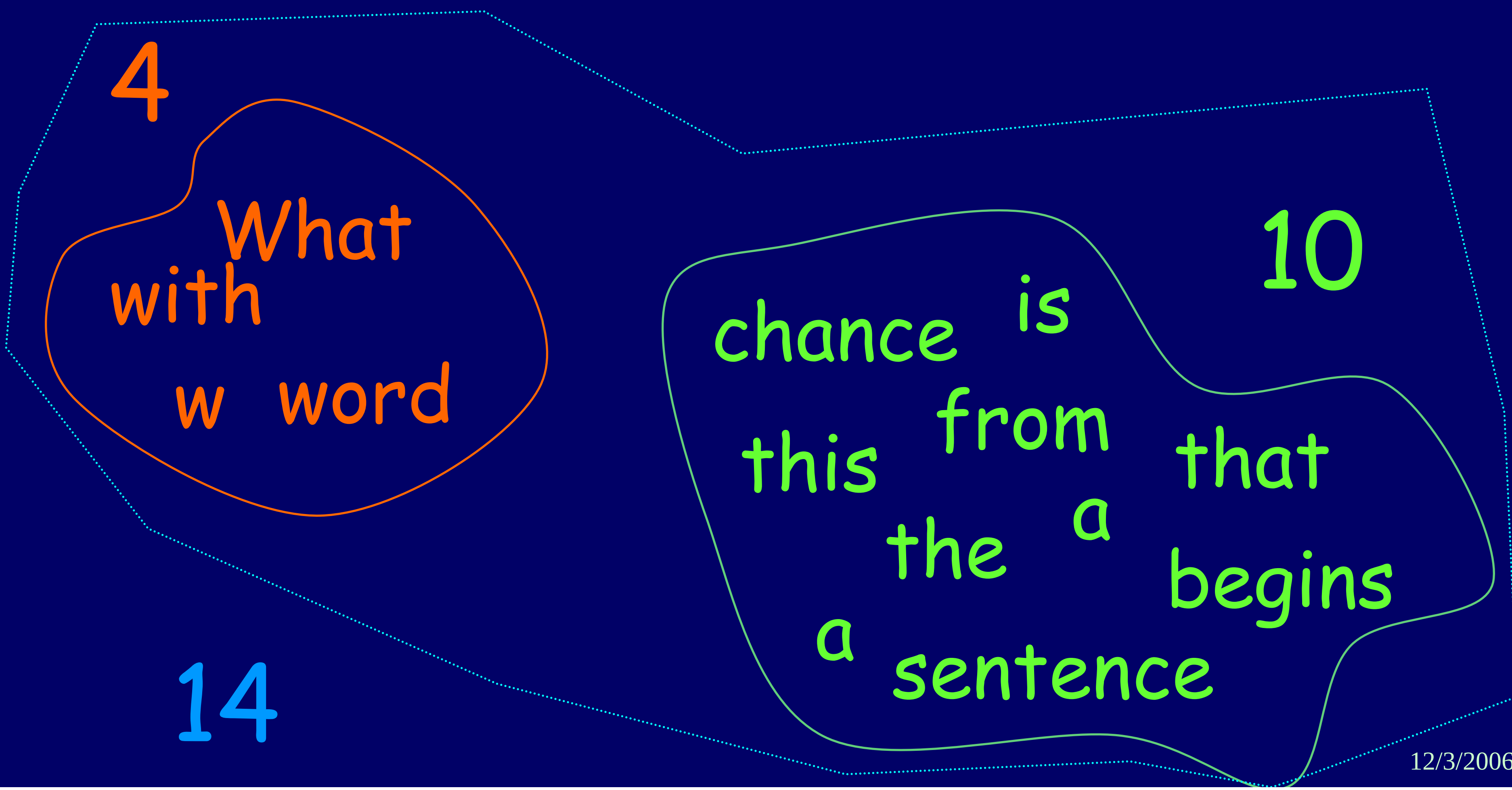
There are 7  & 13 



What is the chance that a person selected at random from the class is a woman ?

Probability as Proportion

What is the chance that a word from this sentence begins with a w?



Space of outcomes

The space of all possible outcomes will be denoted by Ω.

Probability as Proportion

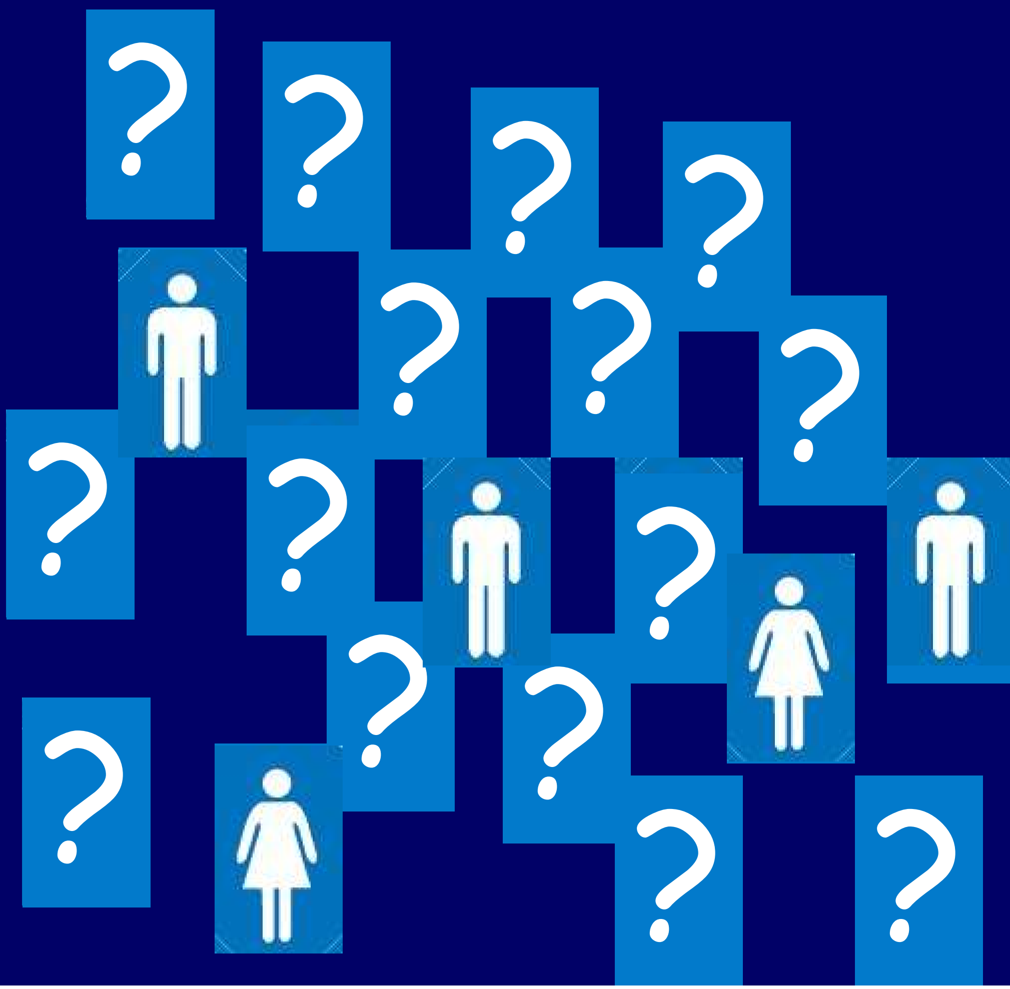
If Ω is finite and each outcome is equally likely than the probability of an event A is:

$$P(A) = \frac{\# A}{\# \Omega}$$

Probability as Proportion

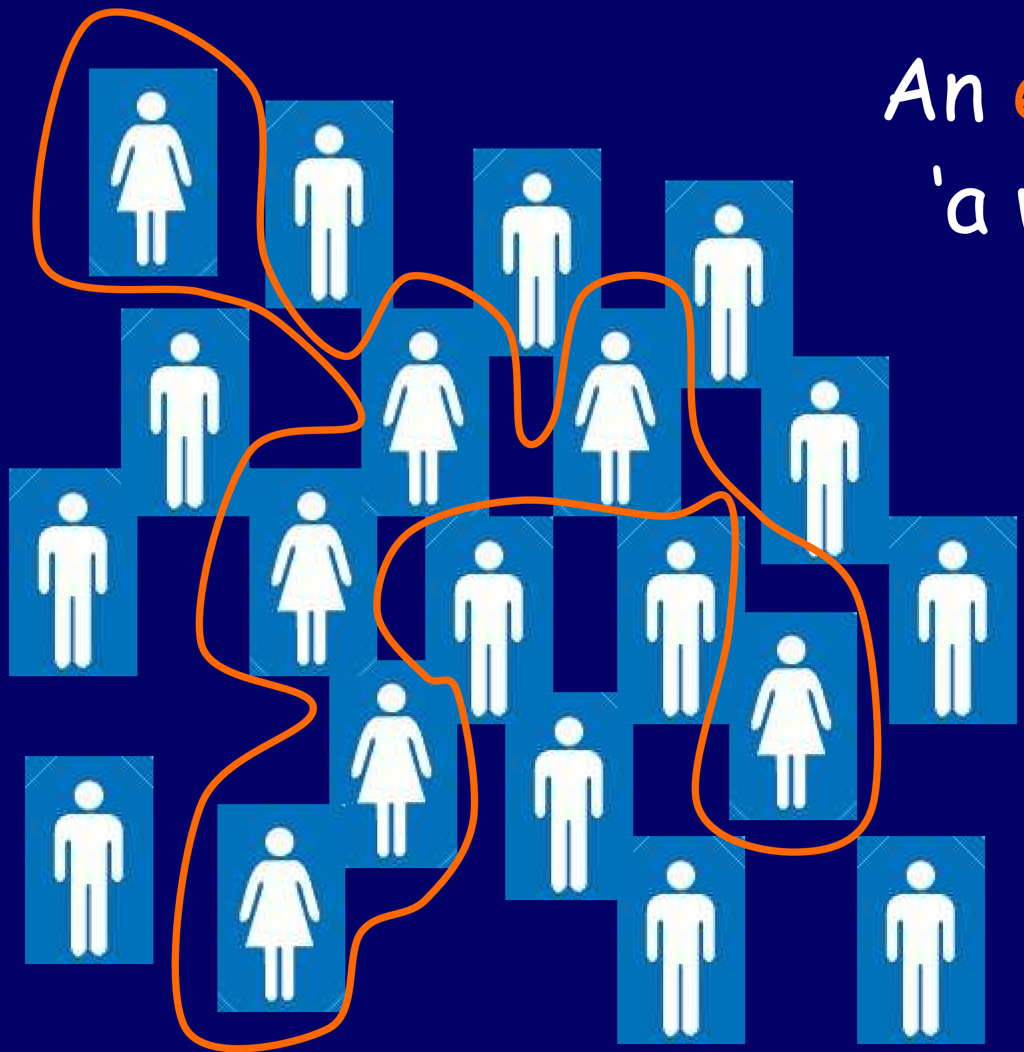
Suppose there are 20 people taking Stat 134.

There are 7  & 13 



What is the chance that a person selected at random from the class is a woman ?

The state space Ω is the entire class.
An outcome is a particular individual.



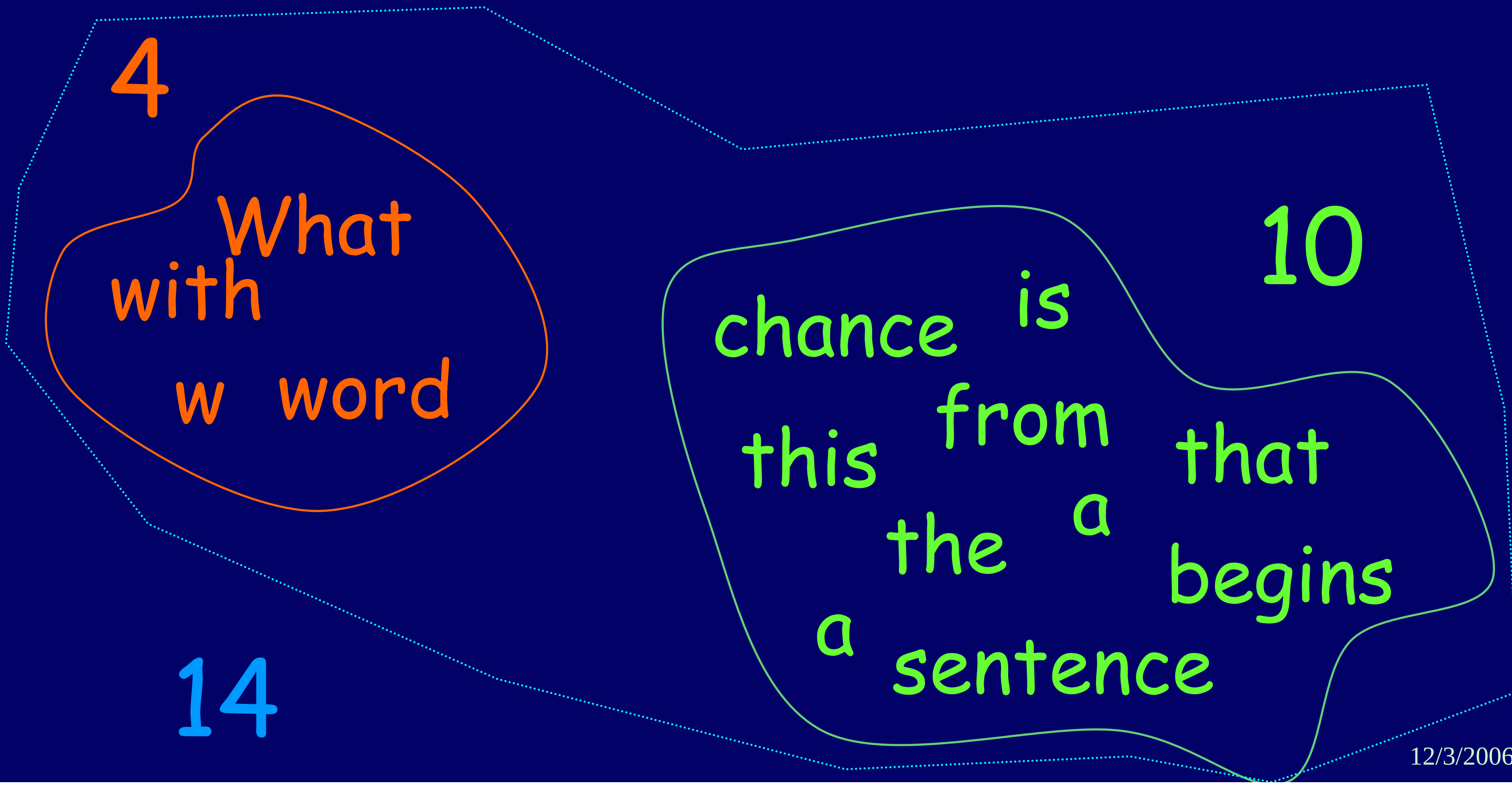
An event:
'a woman is picked' corresponds to

A - the subset of all women in the class.

$P(A) = 7/20$

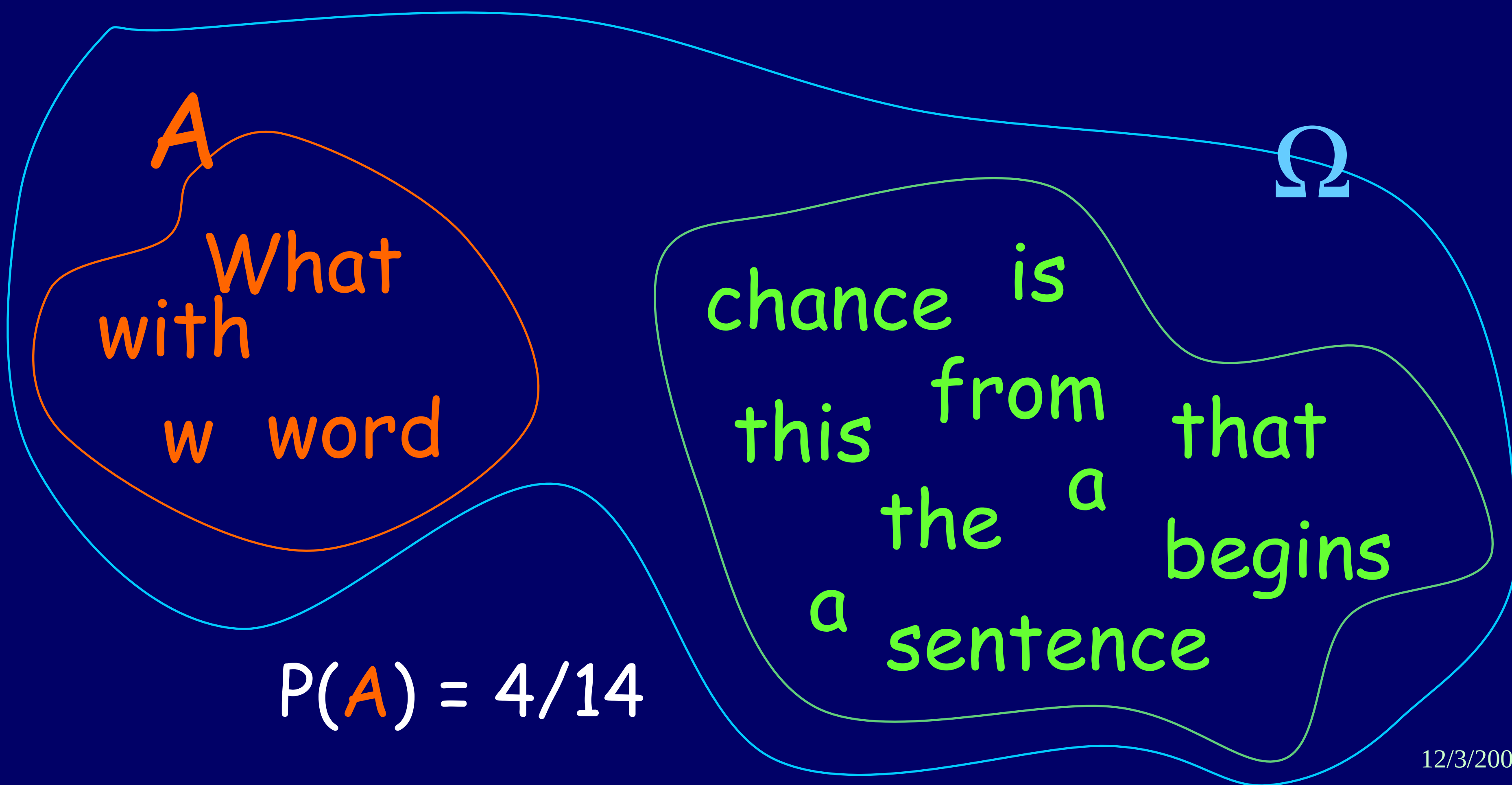
Probability as Proportion

What is the chance that a word from this sentence begins with a w?



The state space Ω is the (set of words of the) sentence. An outcome is a particular word.

An event: ‘a word starts with w’ corresponds to set A of all the words starting with w.



Roll two dice.
What is the chance that the sum is:
Equal to 7?
Equal to 2?
Even?
Odd?

First we need a Sample Space.

Is Ω = {2,3,4,5,6,7,8,9,10,11,12} ?

Problem: are events in Ω equally likely?

Roll two dice.

What is the chance that the sum is:

Equal to 7?

Equal to 2?

Even?

Odd?

The correct sample space Ω:

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Roll two dice.

What is the chance that the sum is:

Equal to 7?

Equal to 2?

Even?

Odd?

6/36 = 1/6

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Roll two dice.

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Even?

Odd?

1/36

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
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Roll two dice.

What is the chance that the sum is:

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Even?

Odd?

18/36 = 1/2

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
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Roll two dice.

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Equal to 7?

Equal to 2?

Even?

Odd?

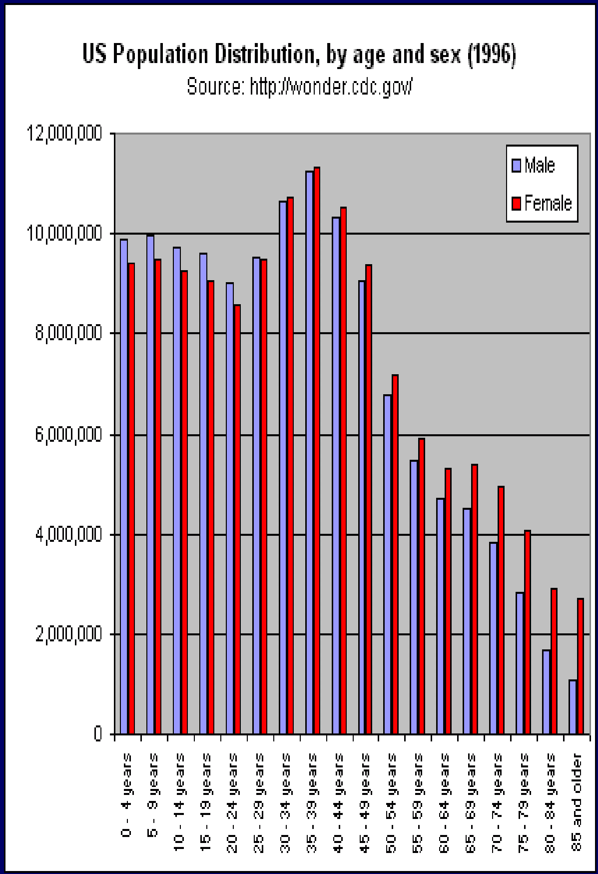
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

18/36 = 1/2

Probability interpretation: as frequency in population.

- In a <u>population</u> we interpret the probability of an event as the proportion of the event in the population.

- The <u>census</u> is based on the assumption that if we take a large enough <u>sample</u> then the observed frequency of an event at the sample should be close to the probability of the event in the population.



Probability Interpretation as frequency in repeated experiments

- Repeating the same experiment over and over again, the observed frequency of experiments ending at an event A should be close to $P(A)$.

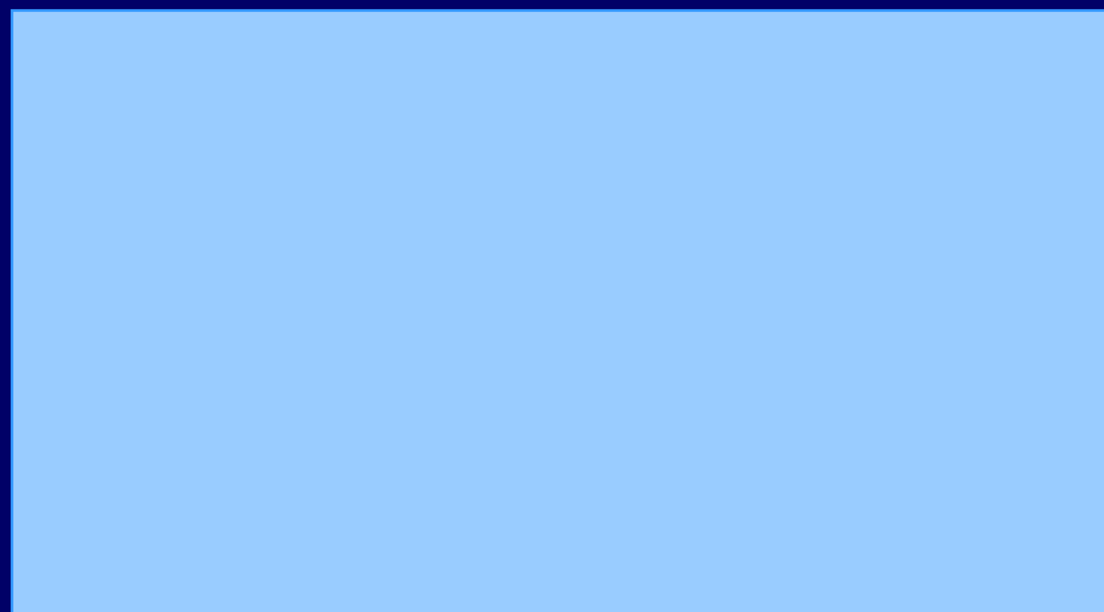
- The fact that the observed frequencies converge to $P(A)$ is called the law of large numbers. We will prove this law later.

Events & Subsets

event: a possible outcome

subset: outcome space Ω

Venn diagram



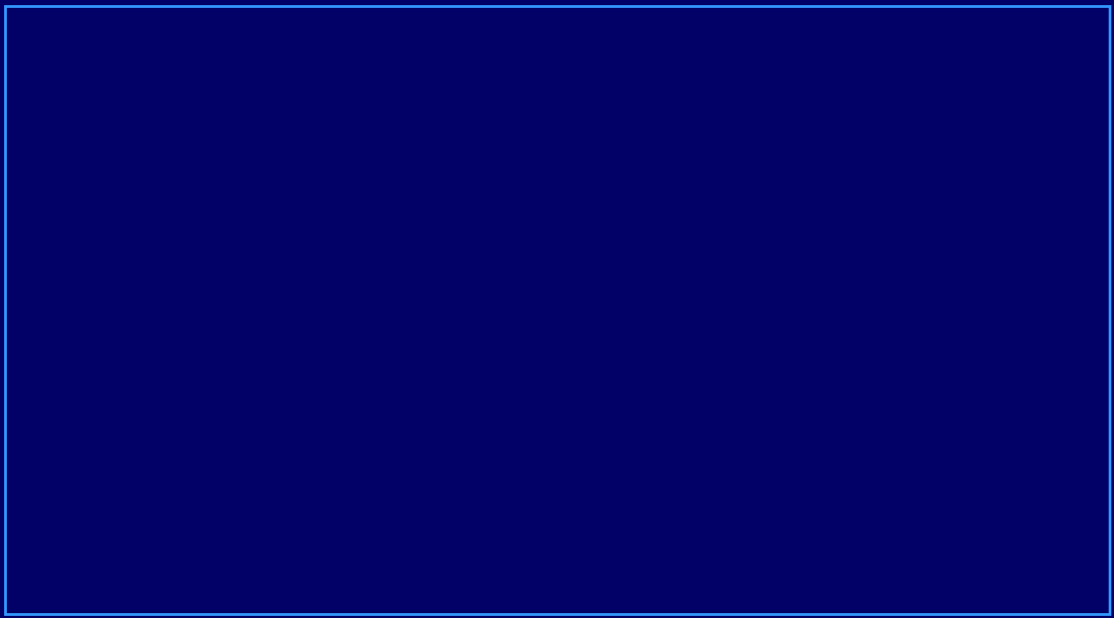
$P(\Omega) = 1$

Events & Subsets

event: impossible outcome –

subset: empty set $\emptyset$

Venn diagram



$P(\emptyset) = 0$

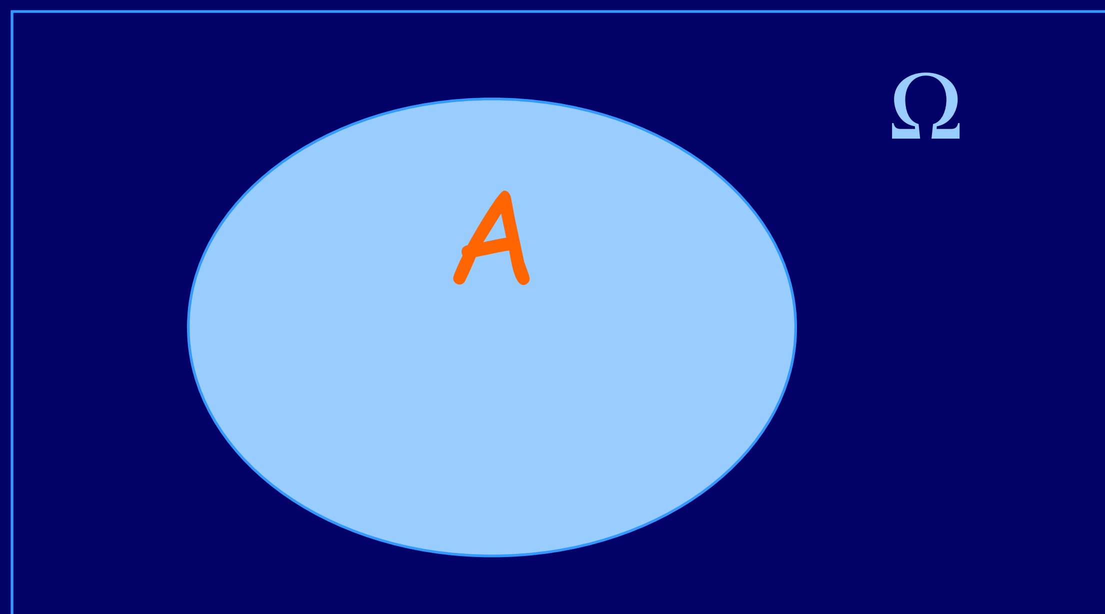
Events & Subsets

event: outcome belongs to A –

– subset: A of Ω

Venn diagram

$0 \leq P(A) \leq 1$

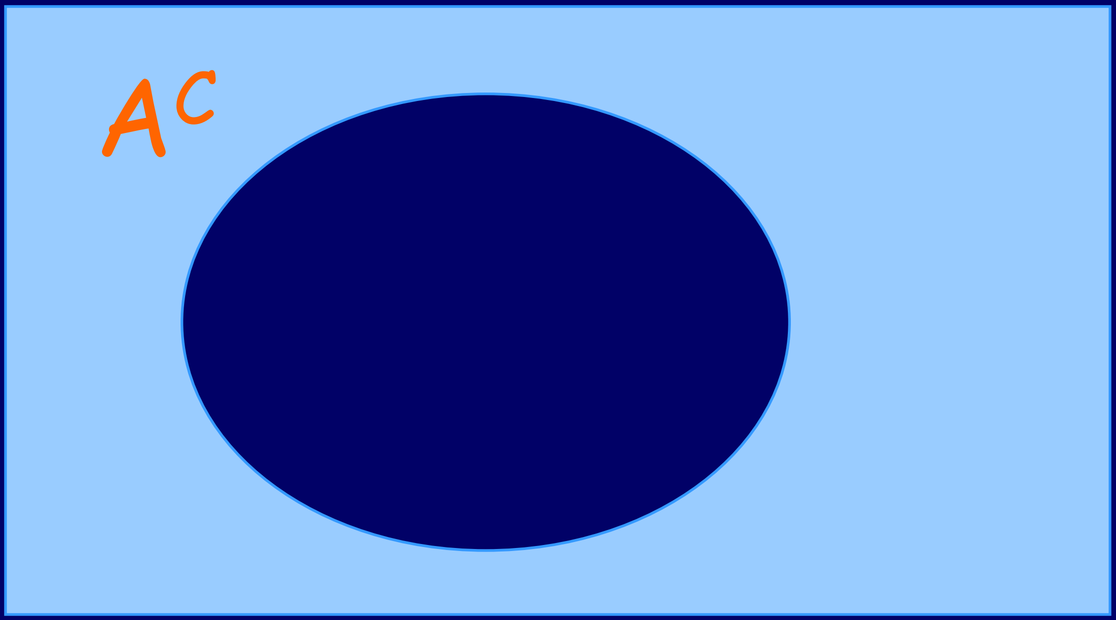


Events & Subsets

event: outcome is not in A –

subset: complement of A in Ω

Venn diagram

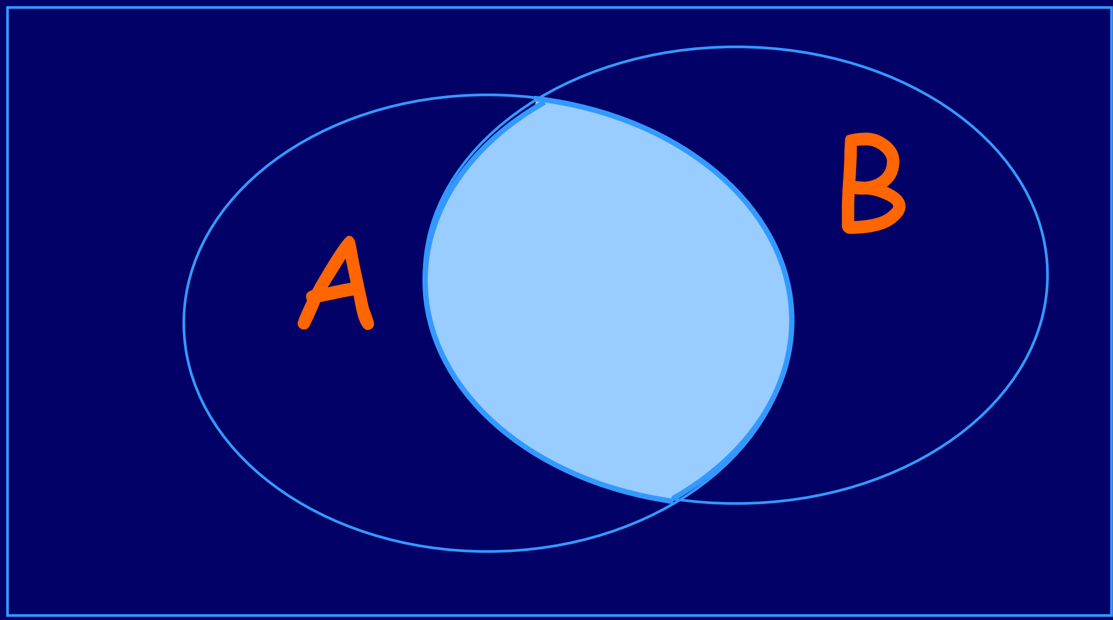


$P(A^c) = 1 - P(A)$

Events & Subsets

event: outcome belongs to A and B – subset: $A \cap B$ of Ω

Venn diagram



$P(A \cap B) \geq 0$

$P(A \cap B) \leq P(A)$

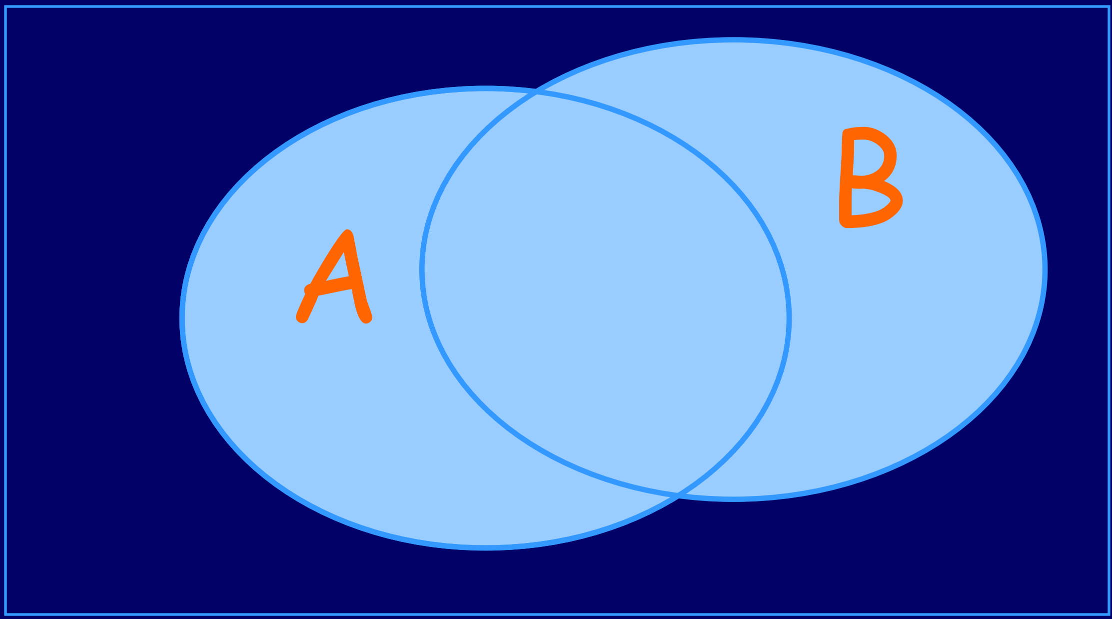
$P(A \cap B) \leq P(B)$

Events & Subsets

event: outcome belongs to A or B –

subset: $A \cup B$ of Ω

Venn diagram



$$P(A \cup B) \geq P(A)$$

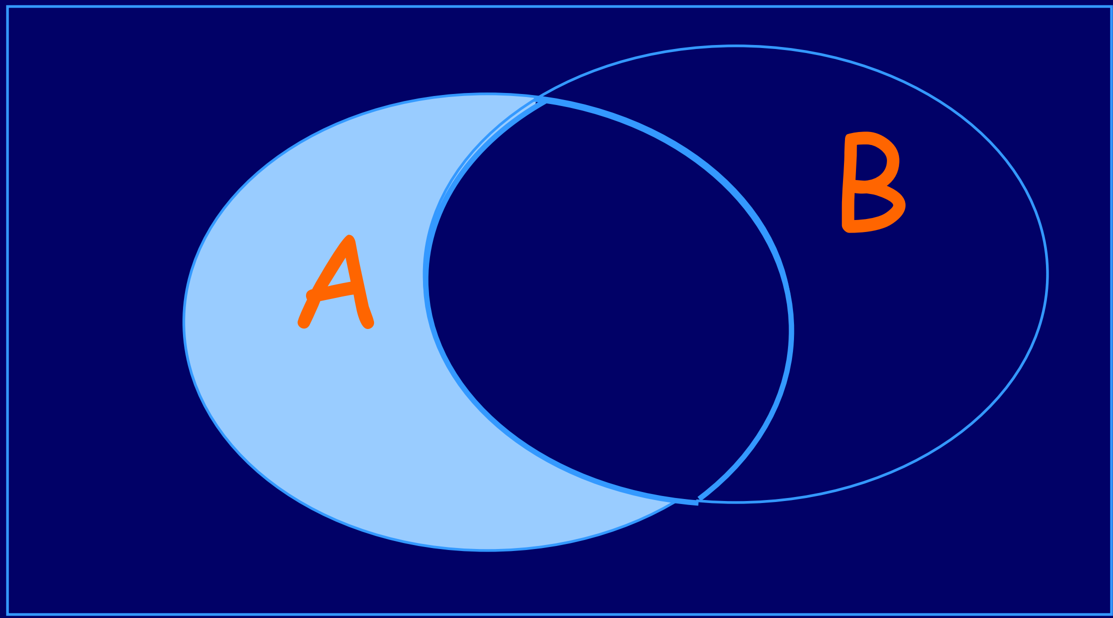
$$P(A \cup B) \geq P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Events & Subsets

event: outcome belongs to A but not to B –
subset: $A \setminus B$ of Ω

Venn diagram



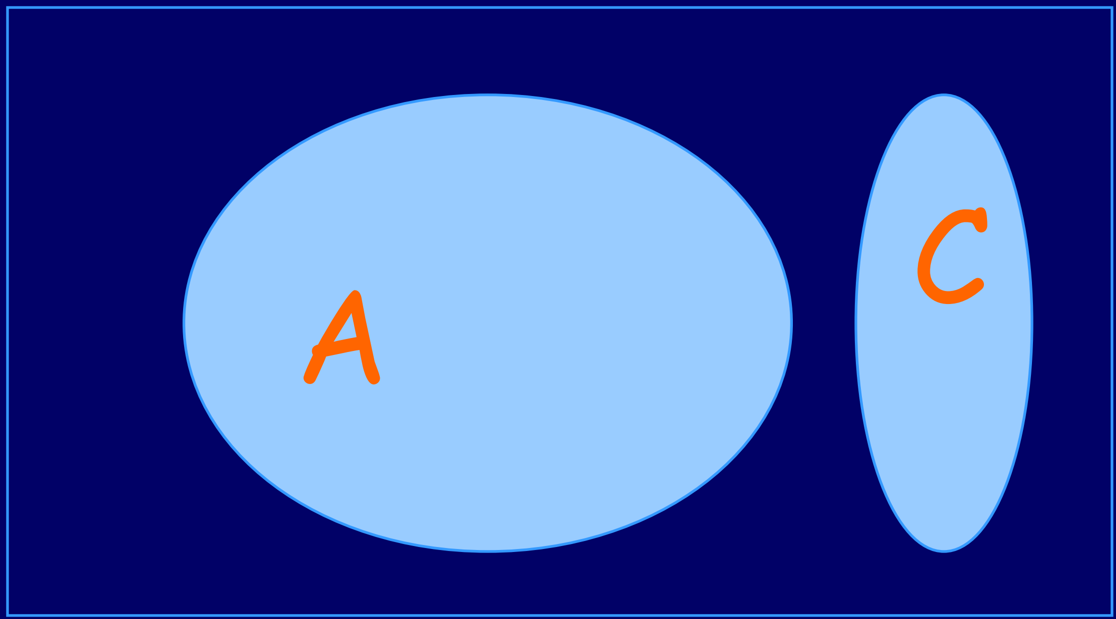
$P(A \setminus B) =$
$P(A) - P(A \cap B)$

Events & Subsets

event: outcome is in A or C with A and C mutually exclusive

subset: disjoint union of A and C

Venn diagram



$P(A \sqcup C) = P(A) + P(C)$

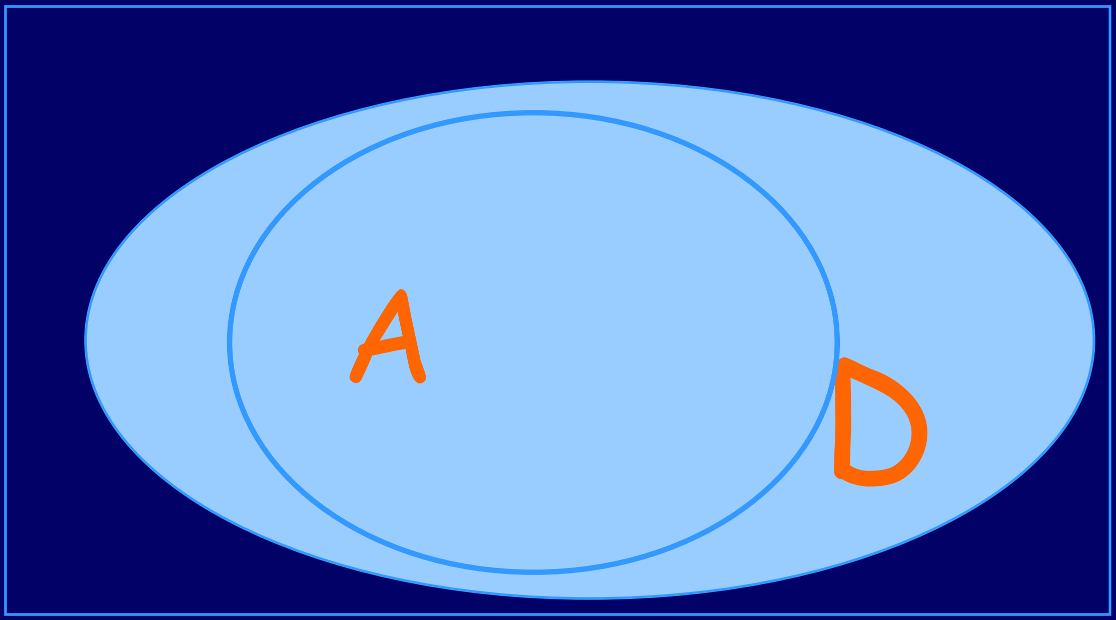
Note: $A \cap C = \emptyset$

Events & Subsets

Event interpretation : if an outcome is in A then it must be in D

Subset interpretation: $A \subset D$

Venn diagram



$P(A) \leq P(D)$

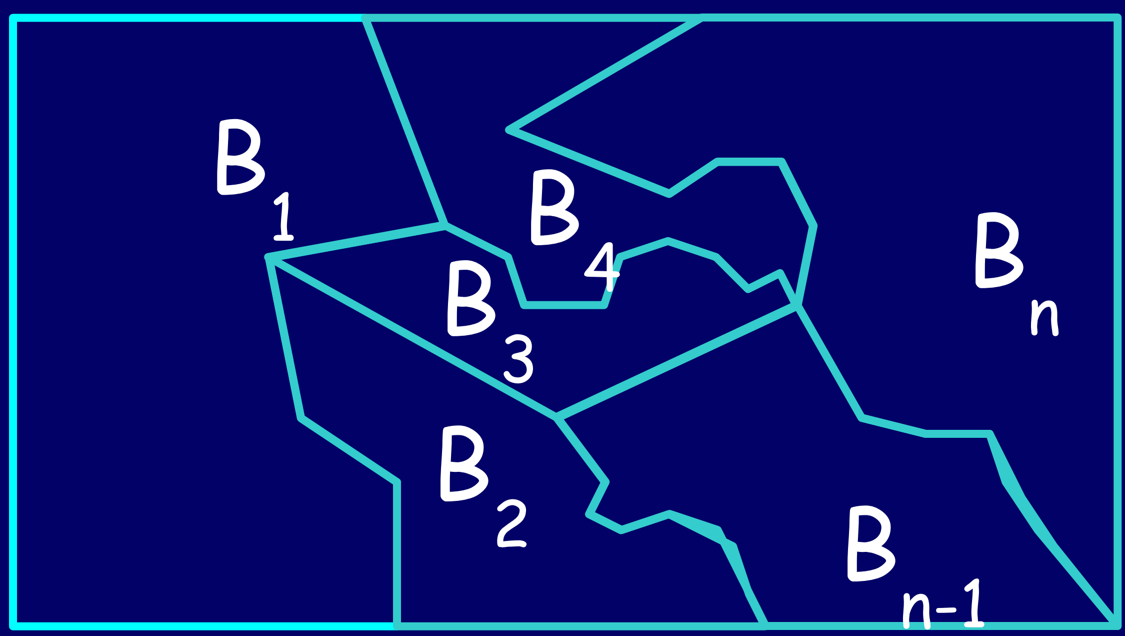
Note: $A \cap D = A$

Partition

If $B_1 \sqcup B_2 \sqcup \ldots \sqcup B_n = B$

We say that B is partitioned into n mutually exclusive events

$B_1, B_2, \ldots, B_n.$



Rules of Probability

- Non-negative:

  $P(B) \geq 0$ for all $B \subseteq \Omega$.

- Additive:

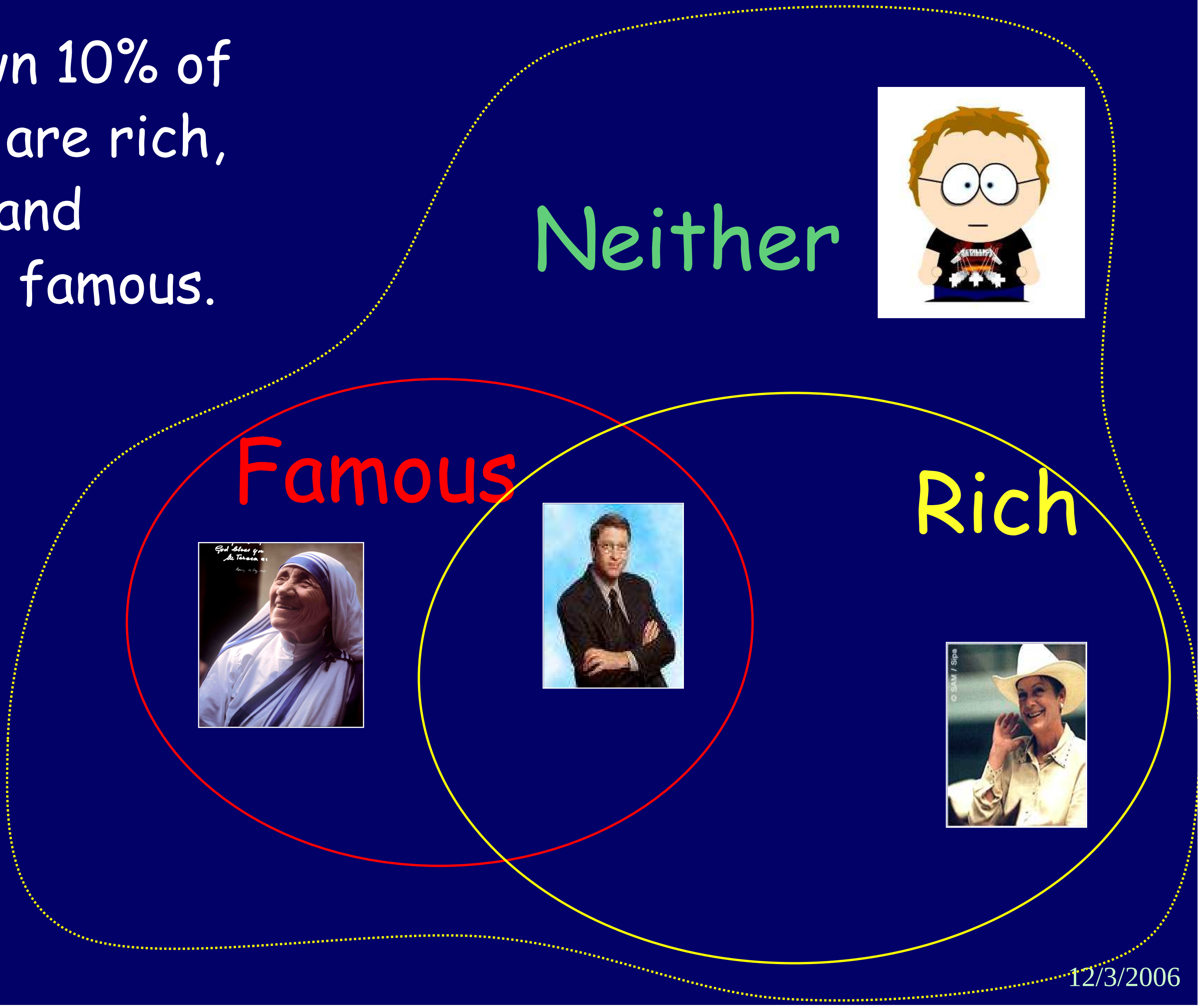
  if $B = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_n$ then
  $P(B) = P(B_1) + P(B_2) + \ldots + P(B_n)$.

- Sums to 1: $P(\Omega) = 1$

A ***distribution*** over Ω is a function P on subsets of Ω which satisfies these three rules.

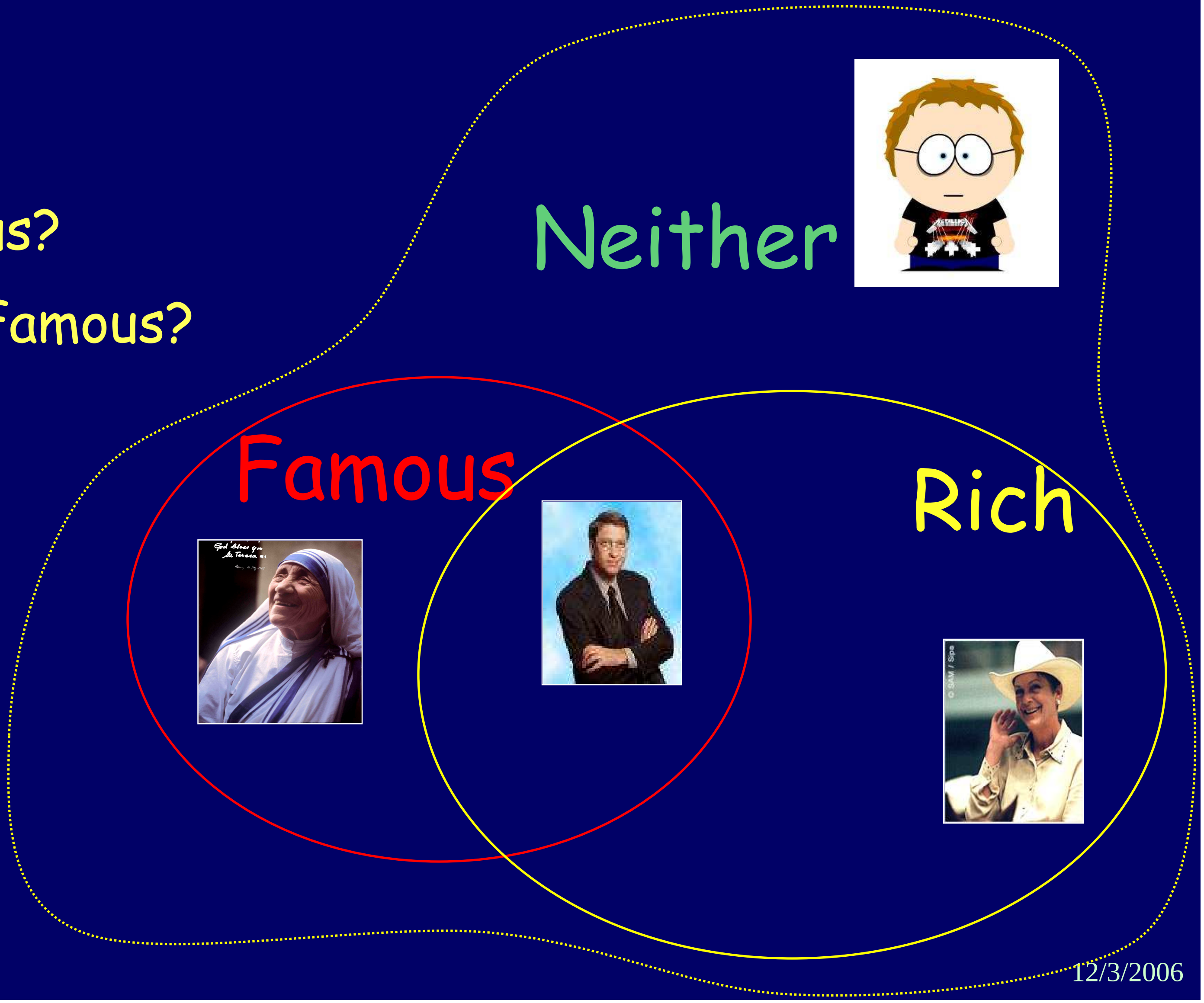
Example: Rich & Famous

In a certain town 10% of the inhabitants are rich, 5% are famous and 3% are rich and famous.



If a town's person is picked at random, what is the chance that he or she is

- Not Rich?
- Rich or Famous?
- Rich but not Famous?



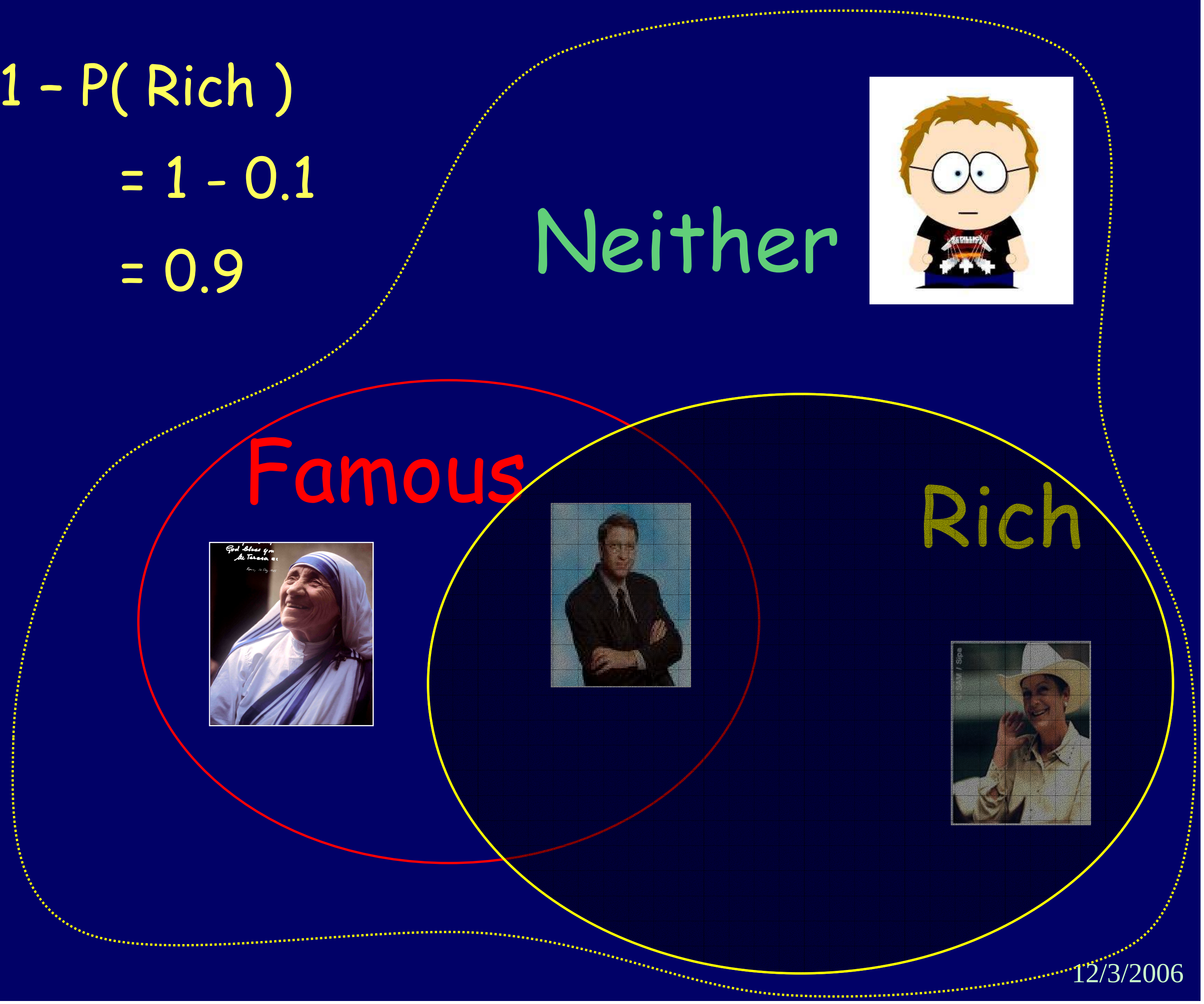
R 10%, F 5%, R&F 3%

Example: Rich & Famous

P(Not Rich)= 1 – P(Rich)

= 1 - 0.1

= 0.9



R 10%, F 5%, R&F 3%

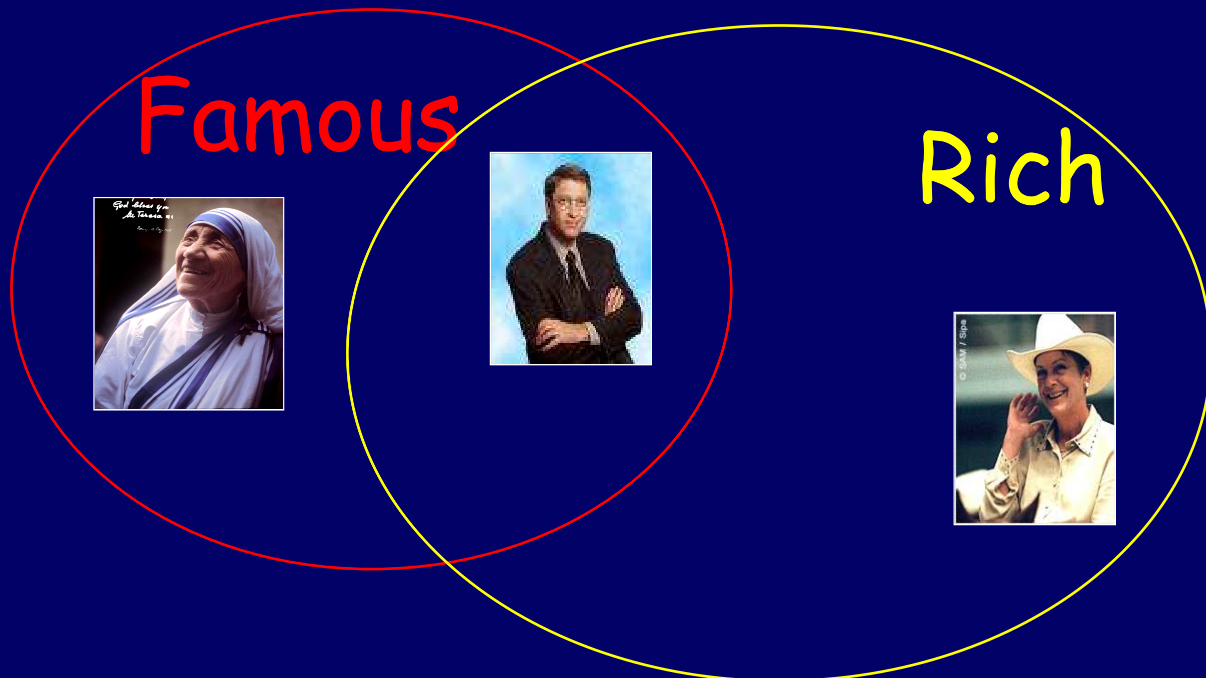
Example: Rich & Famous

P(Rich or Famous)

= P(Rich) + P (Famous) - P(Rich & Famous)

= 0.1 + 0.05 - 0.03

= 0.12



R 10%, F 5%, R&F 3%

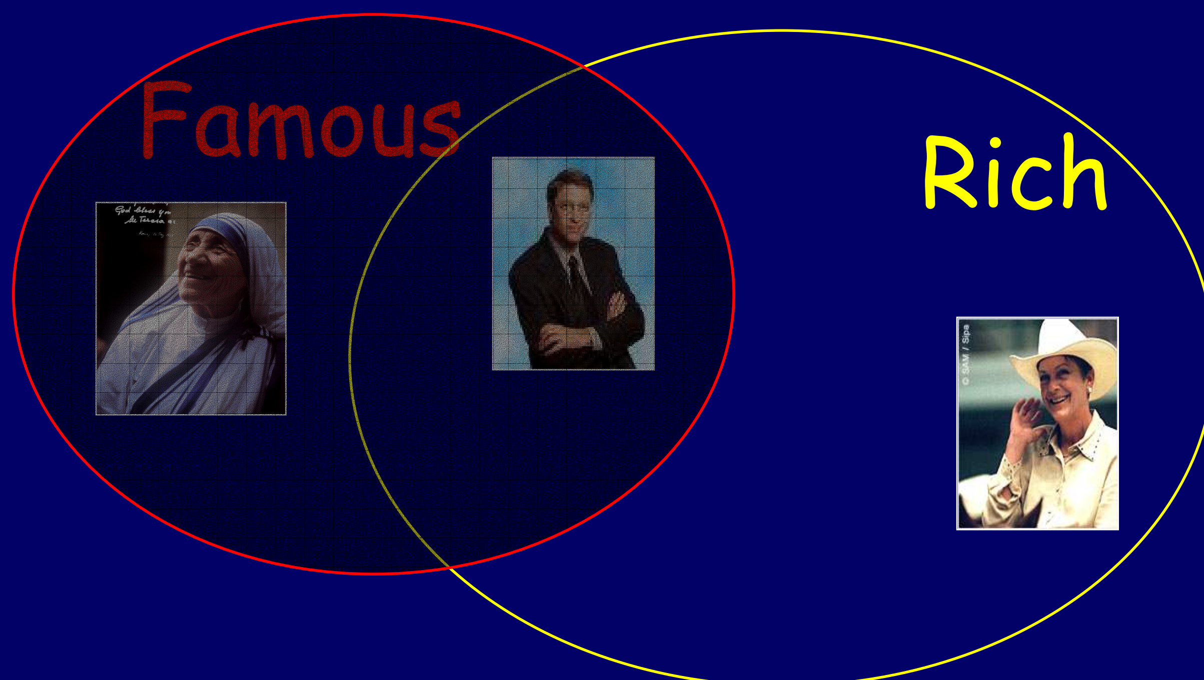
Example: Rich & Famous

P(Rich but not famous)

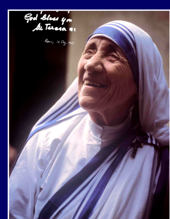
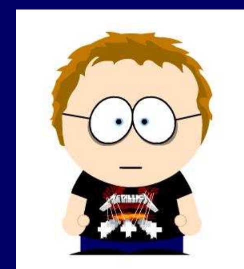
= P(Rich) – P(Rich &Famous)

= 0.1-0.03

= 0.07



Similar computations enable us to complete the following distribution table:

Not Rich Famous	Rich Famous	Rich Not Famous	Not Rich Not Famous
			
0.02	0.03	0.07	0.88

We can use this table to construct a histogram…

Histogram

Is a graphical representation of a distribution.

In a histogram the probability of an event is represented by its area.

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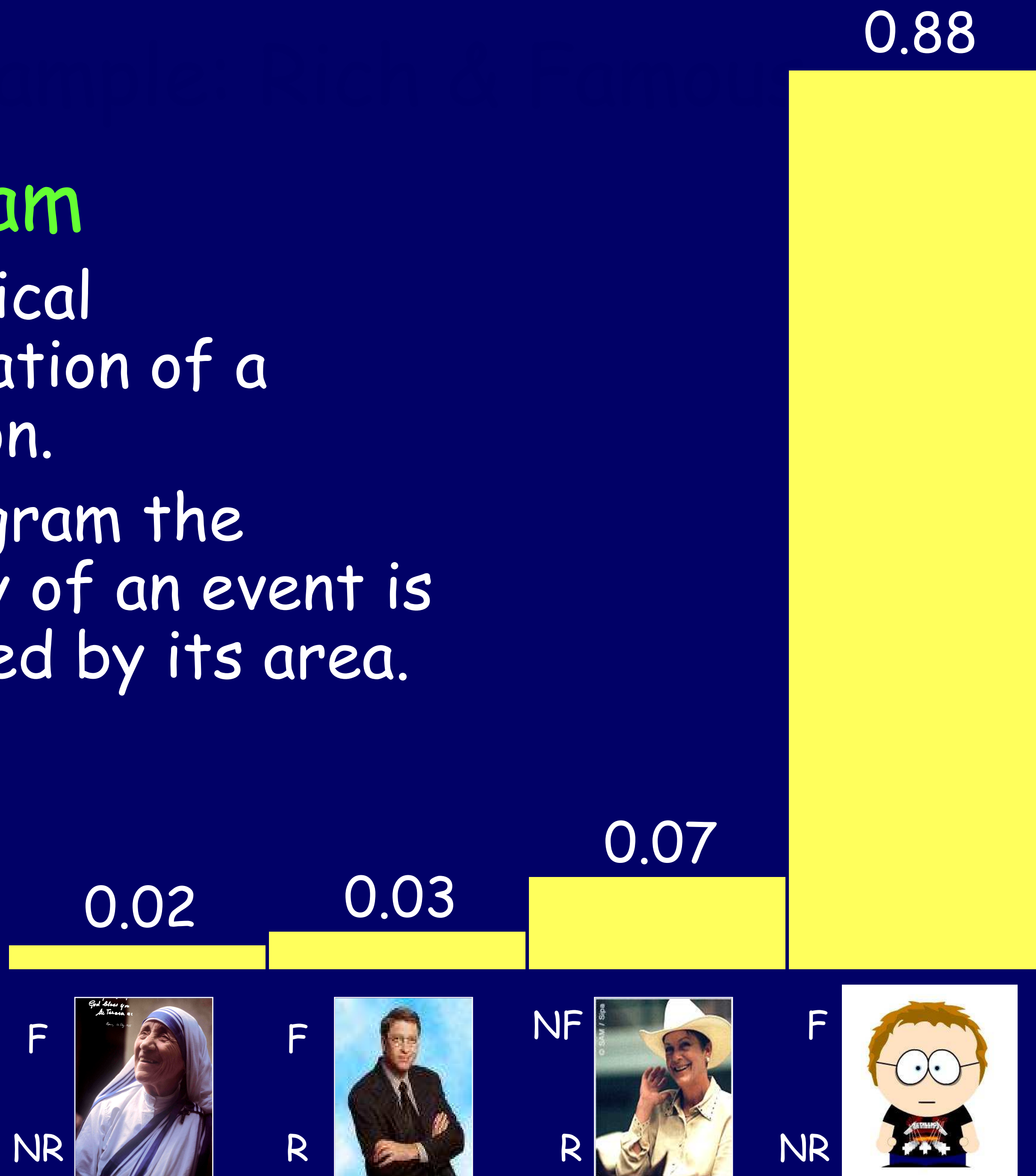
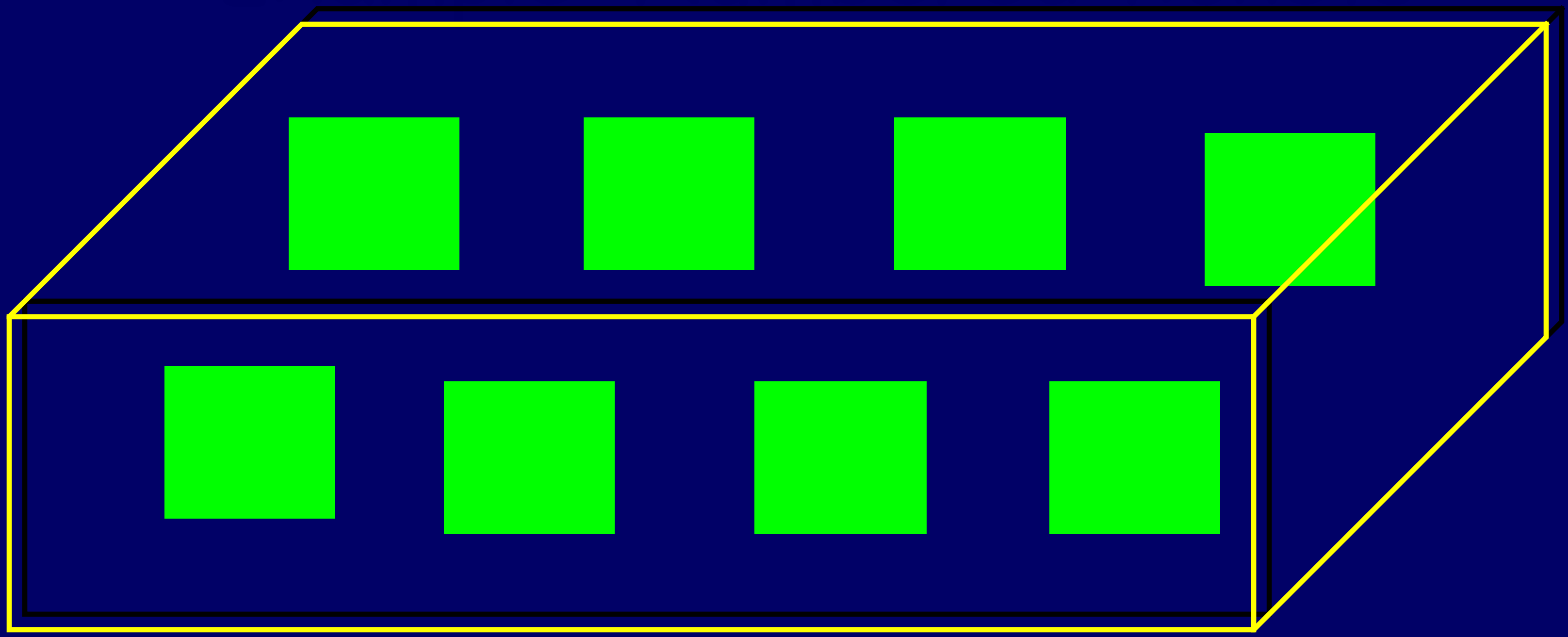


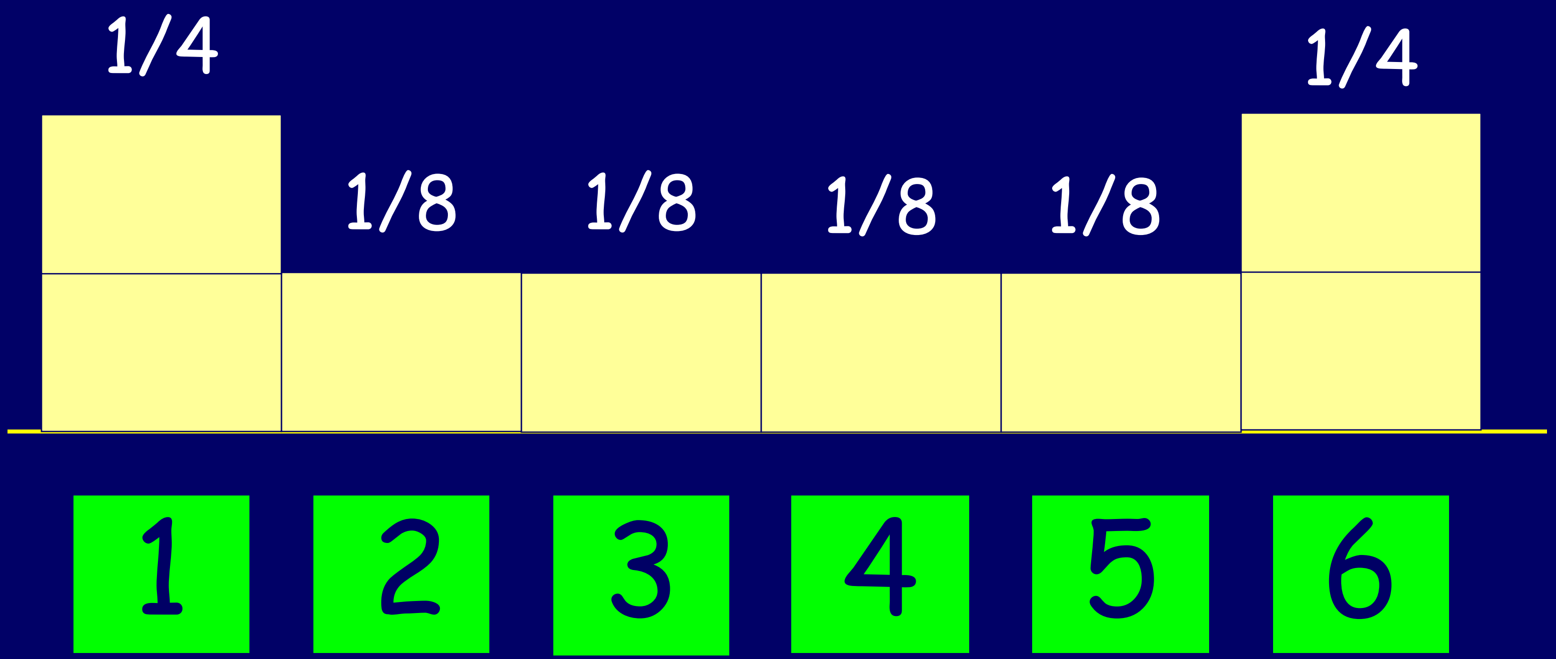


Draw one ticket uniformly at random. What is the chance that the number is greater than 3?

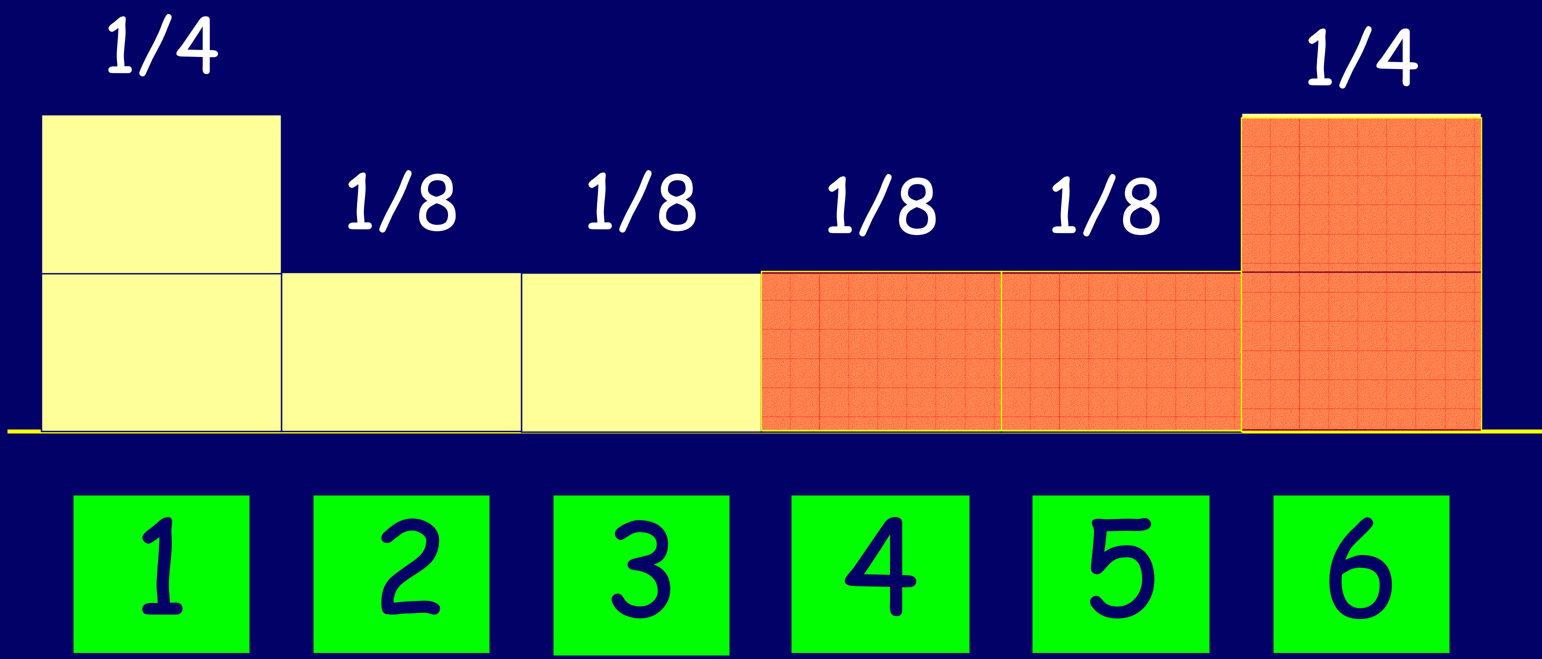
Here is the distribution table:

1	2	3	4	5	6
$\frac{1}{4}$	1/8	1/8	1/8	1/8	$\frac{1}{4}$

Now let's build a histogram:



Let's find the chance that the number on the ticket is greater than 3:

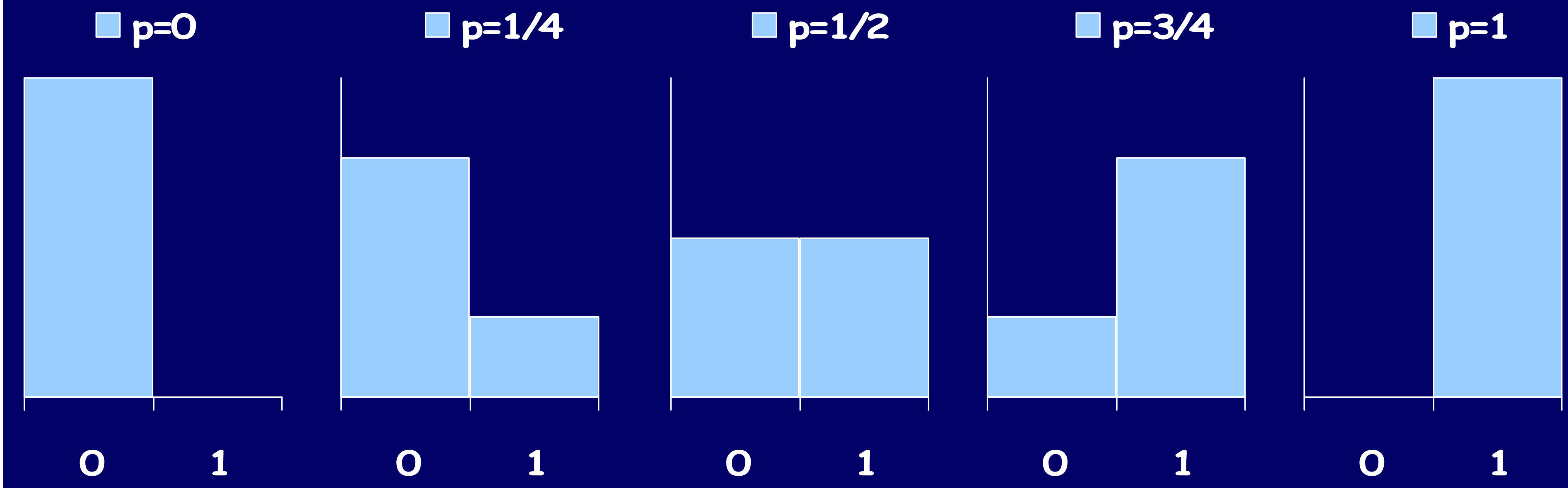


1/8 + 1/8 + 1/4 = 1/2

Named Distributions

Bernoulli (p) Distribution:

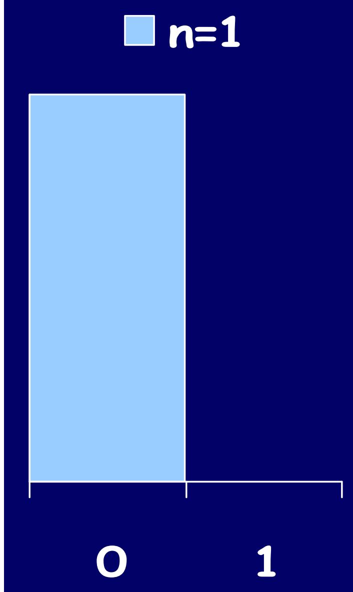
possible outcomes	0	1
probability	1-p	p

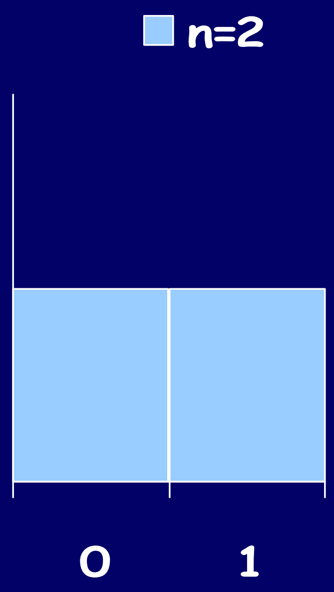


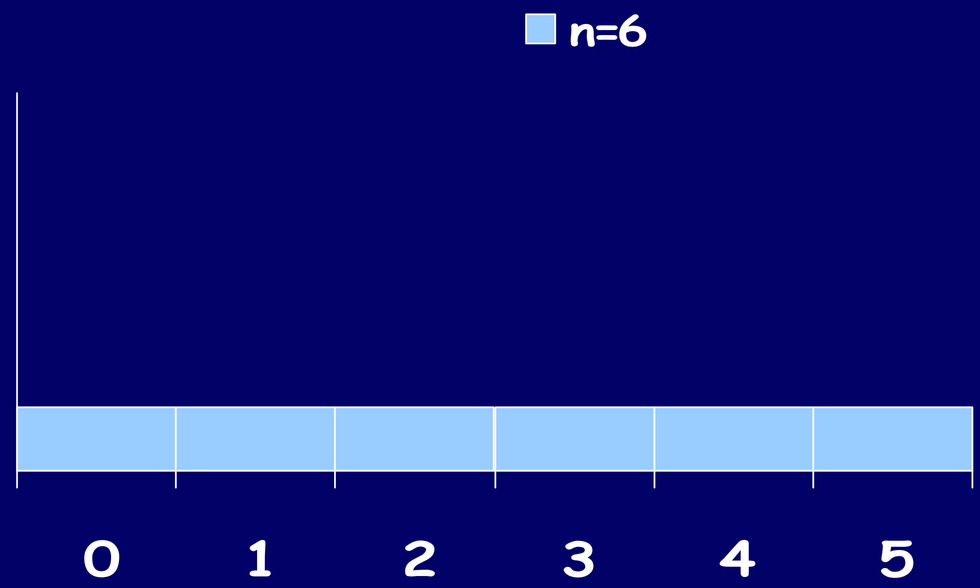
Named Distributions

Uniform Distribution: on {0,1,2,…,n-1}.

possible outcomes	0	1	2	…	n-1
probability	1/n	1/n	1/n		1/n



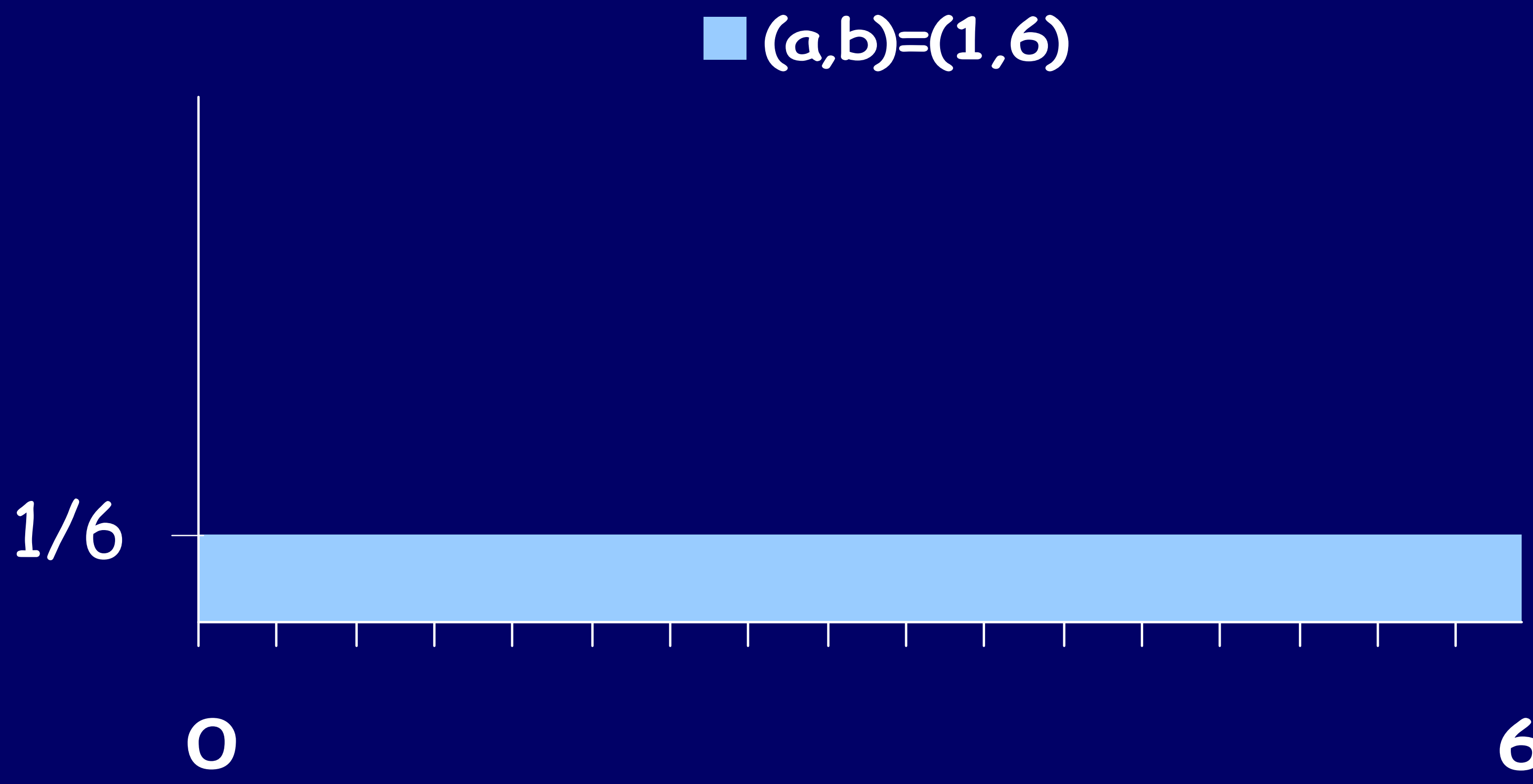




Named Distributions

Uniform (a,b) Distribution:

for $a \leq x < y \leq b$,
p(point in (x,y)) = (y-x)/(b-a)



Named Distributions

Uniform (a,b) Distribution: *for $(x,y) \in B \subset (a,b) \times (c,d)$*
p(point in (x,y)) = Area(B)/((b-a)(d-c)

(a,b)x(c,d) = (0,6)x(0,1)