Introduction to probability

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Follows Jim Pitman’s book:
Probability
Section 4.3
Change of Variables

We discuss the following problem:

• Suppose a continuous random variable $X$ has a density $f_X(x)$.

• If $Y = g(X)$, what’s the density $f_Y(y)$?
Change of Variables

Suppose a continuous random variable $X$ has a density $f_X(x)$. If $Y=g(X)$, what’s the density $f_Y(y)$?

Example (Scaling): $X \sim \text{Uniform}(0,1)$ and $g(X) = aX$.

- $P(0<X<1) = 1$ so $P(0<Y/a <1) = 1$ and $P(0<Y<a) = 1$.
- The interval where $f_Y$ is non-zero is of length $a$.
- The value of $f_Y$ in the interval is $1/a$.
- The total area of under $f_Y$ is 1.
Example: Scaling a uniform

Suppose a continuous random variable $X$ has a density $f_X(x)$. If $Y=g(X)$, what’s the density $f_Y(y)$?

Example (Shift): $X \sim \text{Uniform}(0,1)$ and $g(X) = X+b$.

- $P(0<X<1) = 1$ so $P(0<Y-b<1) = 1$ and $P(b<Y<1+b) = 1$.
- The length of the interval where $f_Y$ is non-zero is still 1.
- The endpoints of the interval where $f_Y$ is non-zero are shifted by $b$.
- The total area under $f_Y$ is 1.
Linear Change of Variables for Densities

**Claim:** If $Y=aX + b$, and $X$ has density $f_X(x)$ then

$$f_Y(y) = \frac{1}{|a|} f_X \left( \frac{y-b}{a} \right).$$

**Example:** suppose $X \sim N(0,1)$ and $Y=aX + b$, then

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad f_Y(y) = \frac{1}{|a| \sqrt{2\pi}} e^{-\frac{(y-b)^2}{2a^2}}.$$ 

This is the calculation of the density of $N(a,b)$ we omitted on section 4.1.
**1-1 functions**

Definition: A function $g(x)$ is 1-1 on an interval $(a,b)$ if for all $x,y$ in $(a,b)$, if $g(x) = g(y)$ then $x = y$.

- In other words, the graph of $g$ cannot cross any horizontal line more than once.

- This implies that $g^{-1}$ is a well defined function on the interval $(g(a), g(b))$ or $(g(b), g(a))$. 
1-1 means Monotonic

Claim: a continuous 1-1 function has to be strictly monotonic, either increasing or decreasing

Pf: If a continuous function $g(x)$ is not strictly monotonic then there exist $x_1 < x_2 < x_3$ such that $g(x_1) \cdot g(x_2) \geq g(x_3)$ or $g(x_1) \geq g(x_2)^2 \cdot g(x_3)$.

This implies by the mean-value theorem that the function cannot be 1-1.
Suppose $Y = g(X)$, where $g$ is 1 to 1 and differentiable.

Then:

$$P[X \in (x, x + dx)] = P[Y \in (y, y + dy)]$$

$$f_X(x) dx = f_Y(y) dy = f_Y(y) |g'(x)| dx.$$ 

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} = \frac{f_X(x)}{|dy/dx|}$$ 

$$P(y < Y \leq y + dy)$$

$$dy \approx g'(x) dx$$
Change of Variables Formula for 1-1 Differentiable Functions

Claim: Let $X$ be a random variable with density $f_X(x)$ on the range $(a,b)$. Let $Y = g(X)$ where $g$ is a 1-1 function on $(a,b)$ Then the density of $Y$ on the interval $(g(a),g(b))$ is:

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} = \frac{f_X(g^{-1}(y))}{\left|\frac{dy}{dx}\right|}$$
Exponential function of an Exponential Variable

**Example:** Let $X \sim \text{Exp}(1) ; \ f_X(x) = e^{-x}, \ (x>0)$. 

Find the density of $Y = e^{-X}$.

**Sol:** We have: $\frac{dy}{dx} = -e^{-x} = -y$ and $x = -\log y$. 
The range of the new variable is: $e^{-\infty} = 0$ to $e^{0} = 1$.

\[ f_Y(y) = \frac{f_X(-\log y)}{|-y|} = \frac{e^{\log y}}{y} = 1 \]

So $Y \sim \text{Uniform}(0,1)$.
Log of a Uniform Variable

Example: Let $X \sim \text{Uniform}(0,1)$; $f_X(x) = 1, (0 < x < 1)$.

Find the density of $Y = \frac{-1}{\lambda} \log(X), \lambda > 0$.

Solution: $x = e^{-\lambda y}$ and $dy/dx = -1/ (\lambda x) = -e^{\lambda y}/\lambda$.

The range of $Y$ is: $0$ to $\infty$.

$$f_Y(y) = \frac{f_X(e^{-\lambda y})}{e^{-\lambda y}/\lambda} = \frac{1}{e^{\lambda y}/\lambda} = \lambda e^{-\lambda y}$$

So $Y \sim \text{Exp}(\lambda)$. 
Let $X \sim \text{Uniform}(0,1) ; \ f_X(x) = 1, \ (0<x<1)$.

Find the density of $Y = \sqrt{X}$.

We have: $X = Y^2$ and $\frac{dy}{dx} = 1/ (2y)$.
The range of the new variable is: 0 to 1.

$f_Y(y) = \frac{1}{|1/2y|} = 2y$
Change of Variables Principle

If $X$ has the same distribution as $Y$ then $g(X)$ has the same distribution as $g(Y)$.

**Question:** If $X \sim \text{Uniform}(0,1)$, what’s the distribution of $-(1/\lambda)\log(1-X)$.

**Hint:** Use the change of variables principle and the result of a previous computation.
Many to One Functions

Now:
\[ P(Y \in (y, y + dy)) = \sum_{x: g(x) = y} P(X \in (x, x + dx)) \]

Gives:
\[ f_Y(y) = \sum_{x: g(x) = y} f_x(x) / \left| \frac{dy}{dx} \right| \]
Density of the Square Function

Let $X$ have the density $f_X(x)$. Find the density of $Y = X^2$.

We saw: If $X$ is the root of a $\text{Unif}(0,1)$ RV:

If $Y = X^2$ then the formula above gives:
And $Y \sim \text{Uniform}(0,1)$
Uniform on a Circle

**Problem:** Suppose a point is picked uniformly at random from the perimeter of a unit circle. Find the density of $X$, the $x$-coordinate of the point.

**Solution:** Let $\Theta$ be the random angle as seen on the diagram. Then $\Theta \sim$ Uniform $(-\pi, \pi)$. $f_\Theta = 1/(2\pi)$.

$X = \cos(\Theta)$, a 2-1 function on $(-\pi, \pi)$. The range of $X$ is $(-1,1)$.

$x = \cos \theta$, $\left| \frac{dx}{d\theta} \right| = | - \sin \theta | = \sqrt{1 - x^2}$.

$$f_X(x) = \sum_{\pm \theta} \frac{1}{2\pi \sin \theta} = \frac{1}{\pi \sqrt{1 - x^2}}.$$
Uniform on a Circle

Problem: Find $E(X)$. 

Solution: Observe that the density $f_X(x) = \frac{1}{\pi \sqrt{1 - x^2}}$ is symmetric with respect to 0. So $E(X) = 0$. 

Problem: Find the density of $Y = |X|$. 

Solution: 

$P(Y \in dy) = 2P(X \in dx)$. 

The range of $Y$ is $(0,1)$. 

Problem: Find $E(Y)$. 

Solution: 

$$E(Y) = \frac{2}{\pi} \int_0^1 \frac{y}{\sqrt{1 - y^2}} dy = \left[ \frac{2}{\pi} \sqrt{1 - y^2} \right]_0^1 = \frac{2}{\pi}.$$
Expectation of $g(X)$.

Notice, that it is not necessary to find the density of $Y = g(X)$ in order to find $E(Y)$.

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$  

The equality follows by substitution.

$$y = g(x), \quad dy = g'(x) dx.$$
Uniform on a Sphere

Problem: Suppose a point is picked uniformly at random from the surface of a unit sphere. Let $\Theta$ be the latitude of this point as seen on the diagram. Find $f_\Theta(\theta)$.

Let $Y$ be the vertical coordinate of the point. Find $f_Y(y)$.

Let $Y \sim \text{Uniform}(-1,1)$.