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Follows Jim Pitman’s book:
Probability
Section 4.2
Random Times

Random times are often described by a distribution with a density.

Examples:

• Lifetime of an individual from a given population.
• Time until decay of a radioactive atom.
• Lifetime of a circuit.
• Time until a phone call arrives.
Random Times

• If $T$ is a random time then its range is $[0, \infty)$.

• A random time with the density $f(t)$ satisfies:

$$P(a < T \leq b) = \int_a^b f(t)dt.$$  

• The probability of surviving past time $s$ is called the **Survival Function**:

$$P(T>s) = \int_s^{\infty} f(t)dt.$$  

• Note that The **Survival Function** completely characterizes the distribution:

$$P(a < T \leq b) = P(T>a) - P(T>b).$$
**Exponential Distribution**

**Definition:** A random time T has an **Exponential(λ)** Distribution if it has a probability density
\[ f(t) = \lambda e^{-\lambda t}, \quad (t \geq 0). \]

• Equivalently, for \((a \leq b < \infty)\)
  \[ P(a < T < b) = \int_a^b \lambda e^{-\lambda t} \, dt = e^{-\lambda a} - e^{-\lambda b}. \]

• Equivalently: \(P(T > t) = e^{-\lambda t}, \quad (t \geq 0).\)

**Claim:** If \(T\) has **Exp(λ)** distribution then:
\[ E(T) = \text{SD}(T) = 1/\lambda \]
Exponential densities

\[ f(t) = \lambda e^{-\lambda t}, \quad (t \geq 0). \]
**Memoryless Property**

**Claim:** A positive random variable T has the Exp(\(\lambda\)) distribution iff T has the **memoryless property**:

\[
P(T > t+s | T > t) = P(T > s) , \quad (t,s \geq 0)
\]

**Proof:** If \(T \sim \text{Exp}(\lambda)\), then

\[
P(T > t+s \mid T > t) = \frac{P(T > t+s \& T > t)}{P(T > t)} = \frac{P(T > t+s)}{P(T > t)}
\]

\[
= e^{-\lambda(t+s)}/e^{-\lambda t} = e^{-\lambda s} = P(T > s)
\]

Conversely, if T has the memoryless property, the survival function \(G(t) = P(T > t)\) must solve \(G(t+s) = G(t)G(s)\).

Thus \(L(t) = \log G(t)\) satisfies \(L(t+s) = L(t) + L(s)\)

and \(L\) is decreasing.

This implies \(L(T) = -\lambda t\) or \(G(t) = e^{-\lambda t}\)
Hazard Rate

The constant parameter $\lambda$ can be interpreted as the instantaneous death rate or hazard rate:

$$P(T \leq t + \Delta | T > t) = 1 - P(T > t + \Delta | T > t)$$

$$= 1 - P(T > \Delta), \text{ by the memoryless property}$$

$$= 1 - e^{-\lambda \Delta}$$

$$= 1 - [1 - \lambda \Delta + \frac{1}{2} (\lambda \Delta)^2 - ...]$$

$$\approx \lambda \Delta$$

The approximation holds if $\Delta$ is small (so we can neglect higher order terms).
Exponential & Geometric Distributions

Claim:
Suppose $G$ has a geometric $p$ distribution, then for all $t$ and $\lambda$: $\lim_{p \to 0} P(pG/\lambda > t) = e^{-\lambda t}$.

Proof:
$P(pG/\lambda > t) = P(G > \lambda t/p)$

$= (1-p)^{\left\lfloor \frac{\lambda t}{p} \right\rfloor}$

$\approx (1-p)^{\frac{\lambda t}{p}} \approx e^{-\lambda t}$. 

Let the unit interval be subdivided into $n$ cells of length $1/n$ each occupied with probability $p = \lambda / n$.

Note that the number of occupied cells is $\text{Bin}(n, \lambda/n)$. 

**Relationship to the Poisson Process**

**Question:** What’s the distribution of the successes in an interval \((s, t)\)?

Number of successes in \((t-s)\) n trials is distributed as \( \text{Bin}((t-s)\ n, \lambda/n) \approx \text{Poisson}((t-s)\lambda) \).

**Question:** What is the distribution of \(W_i\)’s, the waiting times between success \(i-1\) and success \(i\).

\[
\text{Time} = (\# \text{ of empty cells} + 1) \times \frac{1}{n} \sim \text{Goem}(\lambda/n) \frac{1}{n} \approx \exp(\lambda).
\]
Two descriptions of Poisson Arrival Process

1. Counts of arrivals:
   - The Number of arrivals in a fixed time interval of length $t$ is a Poisson($\lambda t$) random variable: $P(N(t) = k) = e^{-\lambda t} (\lambda t)^{-k}/k!$
   - # of arrivals on disjoint intervals are indepen.

2. Times between successes:
   - Waiting times $W_i$ between successes are $\text{Exp}(\lambda)$ random variables: $P(W_i > t) = e^{-\lambda t}$
   - The random variables $W_i$ are independent.
Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So the number of calls $N(s,t)$ between the minutes $s$ and $t$ satisfies:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

The waiting time between the $(i-1)^{st}$ and the $i^{th}$ calls satisfies:

$$W_i \sim \text{Exp}(3).$$

**Question:** What’s the probability that no call arrives between $t=0$ and $t=2$?

**Solution:** $N(0,2) \sim \text{Poisson}(6), \quad P(N(0,2) = 0) = e^{-6} = 0.0025.$
Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of \( \lambda = 3 \) per minute.

So for the number of calls between the minutes \( s \) and \( t \), we have:

\[
N(s,t) \sim \text{Poisson}(3(t-s)).
\]

For the waiting time between the \((i-1)^{st}\) and the \(i^{th}\) calls, we have:

\[
W_i \sim \text{Exp}(3).\]

**Question**: What’s the probability that the first call after \( t=0 \) takes more than 2 minute to arrive?

**Solution**: \( W_1 \sim \text{Exp}(3), \quad P(W_1 \geq 2) = e^{-3 \times 2} = 0.0025. \)

**Note**: the answer is the same as in the first question.
Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes $s$ and $t$, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the $i^{th}$ calls, we have:

$$W_i \sim \text{Exp}(3).$$

**Question:** What's the probability that no call arrives between $t=0$ and $t=2$ and at most 4 calls arrive between $t=2$ and $t=3$?

**Solution:** By independence of $N(0,2)$ and $N(2,3)$, this is

$$P(N(0,2) = 0) * P(N(2,3) \leq 4)$$

$$= e^{-6} e^{-3} (1 + 3 + 3^2/2! + 3^3/3! + 3^4/4!) = 0.0020,$$
Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes $s$ and $t$, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the $i^{th}$ calls, we have:

$$W_i \sim \text{Exp}(3).$$

**Question:** What's the probability that the fourth call arrives within 30 seconds of the third?

**Solution:**

$$P(W_4 \leq 0.5) = 1 - P(W_4 > 0.5) = 1 - e^{-3/2} = 0.7769.$$
Poisson Arrival Process

Suppose phone calls are coming into a telephone exchange at an average rate of $\lambda = 3$ per minute.

So for the number of calls between the minutes $s$ and $t$, we have:

$$N(s,t) \sim \text{Poisson}(3(t-s)).$$

For the waiting time between the $(i-1)^{st}$ and the $i^{th}$ calls, we have:

$$W_i \sim \text{Exp}(3).$$

**Question:** What’s the probability that the fifth call takes more than 2 minutes to arrive?

**Solution:**

$$P(W_1 + W_2 + W_3 + W_4 + W_5 > 2) = P(N(0,2) \leq 4)$$

$$P(N(0,2) \leq 4) = e^{-6} (1 + 6 + 6^2/2! + 6^3/3! + 6^4/4!) = 0.2851.$$
Poisson $r^{th}$ Arrival Times

• Let $T_r$ denote the time of the $r^{th}$ arrival in a Poisson process with intensity $\lambda$. The distribution of $T_r$ is called $\text{Gamma}(r, \lambda)$ distribution.

• Note that: $T_r = W_1 + W_2 + \ldots + W_r$, where $W_i$'s are the waiting times between arrivals.

• Density: $P(T_r \in (t, t+dt))/dt = P[N(0,t) = r-1] \times (\text{Rate});$

$$P(T_r \in (t, t + dt)) = e^{-\lambda t} \frac{(\lambda t)^{r-1}}{(r-1)!} \lambda dt$$

• Right tail probability: $P(T_r > dt) = P(N(0,t) \leq r-1);$ \[ P(T_r > t) = \sum_{k=0}^{r-1} e^{-\lambda t} \frac{(\lambda t)^k}{(k)!}. \]

• Mean and SD: $E(T_r) = r/\lambda \quad SD(T_r) = \sqrt{r}/\lambda.$
Gamma Densities for $r=1$ to 10.

Due to the central limit theorem, the gamma$(r, \lambda)$ distribution becomes asymptotically normal as $r \to \infty$. 

Probability density in multiples of $\lambda$. 

Time in multiples of $1/\lambda$. 

Graph showing the Gamma densities for $r=1$ to $10$. The distribution becomes more normal as $r$ increases, as predicted by the central limit theorem.
Mr. D. is expecting a phone call.

Given that up to the time $t$ minutes the call hasn’t come, the chance that the phone call will not arrive in the next $s$ min. is the same as the chance that it hasn’t come in the first $s$ min.

$$P(T>t+s|T>t) = P(T>s),$$

($t, s \geq 0$).