Introduction to probability

Stat 134       FALL 2005
Berkeley

Lectures prepared by:
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Follows Jim Pitman’s book:
Probability
Section 3.1
X is the sum of two dice.

X\left(\begin{array}{c}3\vline&5\end{array}\right)=6

X\left(\begin{array}{c}4\vline&1\end{array}\right)=5

X\left(\begin{array}{c}1\vline&4\end{array}\right)=5

X\left(\begin{array}{c}2\vline&3\end{array}\right)=5
Probability distribution for X.
$Y$ is the number of aces in a poker hand.

$Y(\text{hand 1}) = 1$

$Y(\text{hand 2}) = 0$

$Y(\text{hand 3}) = 2$
probability distribution for $Y$.
Z is the number of heads in 10 coin tosses.

\[
\begin{align*}
Z(\text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5}}}}}}}}})} & = 5 \\
Z(\text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5}}}}}}}}}) & = 5 \\
Z(\text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5}}}}}}}}}) & = 7 \\
Z(\text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5} \text{\rotateleft{5}}}}}}}}}) & = 6
\end{align*}
\]
Probability distribution for $Z$. 

$P(Z=z)$
Random Variables

A Random Variable is a symbol representing a quantity that is determined by an outcome in a probability space.
Random Variables

Note that for every random variable $X$,

$$\sum_x P(X=x) = 1,$$

where $x$ ranges over all possible values that $X$ can take.
Random Variables

Consider \( Z \), the number of heads in 10 coin tosses.

The range of \( Z \) is the set of all possible values that \( Z \) can take:

\[ \{0,1,2,3,4,5,6,7,8,9,10\} \]
Random Variables

What is the domain of $Z$?

In other words, what determines the value that $Z$ takes?

This is $\Omega = \text{the space of outcomes}$. In this case, an outcome is a particular sequence of coin tosses.

$Z(\text{sequence}) = 5$

$\Omega$ has $2^{10} = 1024$ elements.
Several Random Variables

Given two RV $X$ and $Y$ defined on the same outcome space $\Omega$, we can consider a random pair $(X,Y)$. 

$$(X,Y)(\omega) = (X(\omega), Y(\omega)).$$

Then event $(X,Y) = (x,y)$ can be written as:

$$\{\omega : (X,Y)(\omega) = (x,y)\} = \{\omega : X(\omega) = x\} \cap \{\omega : Y(\omega) = y\}.$$
Example:

Let $X$ be the sum of two dice and $Y$ be the minimum.

<table>
<thead>
<tr>
<th>X,Y</th>
<th>2,1</th>
<th>3,1</th>
<th>4,1</th>
<th>5,1</th>
<th>6,1</th>
<th>7,1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,1</td>
<td>4,1</td>
<td>5,2</td>
<td>6,2</td>
<td>7,2</td>
<td>8,2</td>
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<tr>
<td>4,1</td>
<td>5,2</td>
<td>6,3</td>
<td>7,3</td>
<td>8,3</td>
<td>9,3</td>
<td></td>
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<tr>
<td>5,1</td>
<td>6,2</td>
<td>7,3</td>
<td>8,4</td>
<td>9,4</td>
<td>10,4</td>
<td></td>
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<tr>
<td>6,1</td>
<td>7,2</td>
<td>8,3</td>
<td>9,4</td>
<td>10,5</td>
<td>11,5</td>
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</tr>
<tr>
<td>7,1</td>
<td>8,2</td>
<td>9,3</td>
<td>10,4</td>
<td>11,5</td>
<td>12,6</td>
<td></td>
</tr>
</tbody>
</table>
Joint Distributions

The distribution of \((X,Y)\) is called the joint distribution of \(X\) and \(Y\):

\[
P(x,y) = P(X=x, Y=y),
\]

satisfying

\[
P(x,y) \geq 0 \quad \text{and} \quad \sum_{\text{all } (x,y)} P(x,y) = 1.
\]
We can build a probability distribution table for the $X$ and $Y$ in the previous example:
<table>
<thead>
<tr>
<th>P(X,Y)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/36</td>
<td>0</td>
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<td>7</td>
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<td>2/36</td>
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<td>0</td>
<td>1/36</td>
</tr>
</tbody>
</table>
Joint Distributions

Question:
Suppose you know the distribution of X and the distribution of Y separately, does this determine their joint distribution?
Several Random Variables

Example:

\( X \) is the result of the first draw from the box containing 1,2,3,4,5,6;

\&

\( Y \) - the result of the second draw after the first ticket has been replaced.
## Joint Distribution

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1/36</td>
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<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
</tbody>
</table>

$P(X,Y)$
Several Random Variables

Example:

\(X\) is the result of the first draw from the box containing 1, 2, 3, 4, 5, 6;

\&

\(Z\) - the result of the second draw without replacing the first ticket.
### Joint Distribution

<table>
<thead>
<tr>
<th>X</th>
<th>Z</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
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<td>3</td>
<td>1/30</td>
<td>1/30</td>
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<td>1/30</td>
<td>0</td>
<td>1/30</td>
</tr>
</tbody>
</table>

\[ P(X, Z) \]
## Joint Distribution

<table>
<thead>
<tr>
<th>X</th>
<th>Z</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>dist. X (row sums)</th>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>P(X=1)=1/6</td>
</tr>
<tr>
<td>2</td>
<td>1/30</td>
<td>0</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>P(X=2)=1/6</td>
</tr>
<tr>
<td>3</td>
<td>1/30</td>
<td>1/30</td>
<td>0</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>P(X=3)=1/6</td>
</tr>
<tr>
<td>4</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>0</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>P(X=4)=1/6</td>
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<td>5</td>
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<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>0</td>
<td>1/30</td>
<td>1/30</td>
<td>P(X=5)=1/6</td>
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<tr>
<td>6</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>0</td>
<td>1/30</td>
<td>P(X=6)=1/6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>dist. Z (column sums)</th>
<th>P(Z=1)=1/6</th>
<th>P(Z=2)=1/6</th>
<th>P(Z=3)=1/6</th>
<th>P(Z=4)=1/6</th>
<th>P(Z=5)=1/6</th>
<th>P(Z=6)=1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
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<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
<td>1/30</td>
</tr>
</tbody>
</table>
Joint Distributions

Question:
Suppose you know the distribution of $X$ and the distribution of $Y$ separately, does this determine their joint distribution?

Answer:
It does not ...
Marginal Probabilities

\[ P(X=x) = \sum_y P(X=x, Y=y) \]

\[ P(Y=y) = \sum_x P(X=x, Y=y) \]
### Joint Distribution

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>dist. X (row sums)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$P(X=1) = \frac{1}{6}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
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<td>$\frac{1}{36}$</td>
<td>$P(X=2) = \frac{1}{6}$</td>
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<td>3</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
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<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$P(X=3) = \frac{1}{6}$</td>
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<td>4</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
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<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$P(X=4) = \frac{1}{6}$</td>
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<tr>
<td>5</td>
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<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$P(X=5) = \frac{1}{6}$</td>
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<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{36}$</td>
<td>$P(X=6) = \frac{1}{6}$</td>
</tr>
</tbody>
</table>

| dist. Y (column sums) | $P(Z=1) = \frac{1}{6}$ | $P(Z=2) = \frac{1}{6}$ | $P(Z=3) = \frac{1}{6}$ | $P(Z=4) = \frac{1}{6}$ | $P(Z=5) = \frac{1}{6}$ | $P(Z=6) = \frac{1}{6}$ |
Random Variables with the Same Distribution

Random Variables $X$ and $Y$ have the same or identical distribution if they have the same range and for every value $x$ in their range

$$P(X=x) = P(Y=x).$$
Recall:

$X$ is the result of the first draw from the box $\{1,2,3,4,5,6\}$.

$Y$ - the result of the second draw after the first ticket has been replaced;

$Z$ - the result of the second draw without replacing the first ticket.
Draws from a box.

Observe that $X$, $Y$ & $Z$ all have the same distribution: uniform on \{1,2,3,4,5,6\}.

Yet they are not equal as variables:

If $X = 3$ then $Z$ cannot equal 3.

($Y$ and $Z$ are not even defined on the same space of outcomes...
Change of Variables Principle

If \( X \) and \( Y \) have the same distribution then so do \( g(X) \) and \( g(Y) \), for any function \( g \). For example:

\[ X^2 \text{ has the same distribution as } Y^2. \]
Equality of Random Variables

Random variables $X$ and $Y$ defined over the same outcome space are equal, written $X = Y$, iff

$X = Y$ for every outcome in $\Omega$.

In particular, $P(X = Y) = 1$. 
Example: Equal RV's

Suppose we have a box with tiles each having the same area $A$.

Let $X$ be the height of the tiles, $Y$ be their width and $Z = A/X$. Then $Z = Y$ as random variables.
The probability that $X$ and $Y$ satisfy some condition is the sum of $P(x,y)$ over all pairs $(x,y)$ which satisfy that condition.

$$P(X < Y) = \sum_{(x,y): x < y} P(x,y) = \sum_{\text{all } x} \sum_{y:y > x} P(x,y)$$

$$P(X = Y) = \sum_{(x,y): x = y} P(x,y) = \sum_{\text{all } x} P(x,x)$$
Distribution of any function of $X$ and $Y$ can be determined from the joint distribution:

$$P(f(X,Y) = z) = \sum_{(x,y): f(x,y) = z} P(x,y)$$
**Functions of Random Variables**

Let $X,Y$ represent the results of two draws with replacement from a box with $(1, 2,\ldots,6)$.

Let $Z=\max(X,Y)$. We say $Z=f(X,Y)$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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</tbody>
</table>
Conditional Distribution given an event.

For any event $A$, and any random variable $Y$, the collection of conditional probabilities

$$P(Y \in B \mid A) = \frac{P(Y \in B, A)}{P(A)}$$

defines the conditional distribution of $Y$ given $A$. 
Conditional Distribution Given A.

We can compute $P(Y \in B | X \in A)$ by partitioning $B$:

$$P(Y \in B | X \in A) = \frac{P(Y \in B, X \in A)}{\sum_y P(Y = y, X \in A)}.$$

Hence, if $Y$ has a finite range this distribution is specified by $P(Y = y | A)$ for $y \in$ range of $Y$.

We can let $A : X = x$, for any random variable $X$ defined over the same space of outcomes.
**Conditional Distribution of Y Given X=x.**

For every value of x in the range of X, as y varies over the range of Y the probabilities

\[ P(Y=y|X=x) \]

define the **conditional distribution of Y given X=x.**
Rules of Conditional Probability

- Non-negative: \( P(Y \in B \mid A) \geq 0 \) for \( A, B \subseteq \Omega \).

- Additive: if \( B = B_1 \cup B_2 \cup \cdots \cup B_n \)
  \[
  P(Y \in B \mid A) = P(Y \in B_1 \mid A) + P(Y \in B_2 \mid A) + \ldots + P(Y \in B_n \mid A).
  \]

- Sums to 1: \( P(Y \not\in \Omega \mid A) = 1 \)
Multiplication Rule

If the marginal distribution of $X$ and the conditional distribution of $Y$ given $X=x$ are known, then we can find the joint distribution of $X$ and $Y$:

$$P(X=x, Y=y) = P(X=x) P(Y=y|X=x).$$
Independence

Suppose that for any \( x \) and \( y \)

\[ P(Y = y \mid X = x) = P(Y=y). \]

This means that knowing what value \( X \) takes does not influence the probabilities for \( Y \) so values of \( X \) and \( Y \) do not depend on each other.

This property defines independence.
Independence

Now suppose we know the marginal distribution of $X$ and the conditional distribution of $Y$ given $X=x$. Then the joint density is:

$$P(X=x, Y=y) = P(X=x) P(Y=y|X=x).$$

If $X$ and $Y$ are independent then

$$P(X=x, Y=y) = P(X=x) P(Y=y).$$

This is an equivalent definition of independence.
Independence

Suppose we know that for $X$ and $Y$

$$P(X=x, Y=y) = P(X=x) P(Y=y).$$

Then

$$P(Y=y | X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$= \frac{(P(X=x) P(Y=y))}{P(X=x)}$$

$$= P(Y=y).$$
This property does not always hold!

\[ P(X=x, Y=y) \neq P(X=x) \cdot P(Y=y) \]

for general X and Y.
Dependence

Example: draws without replacement from a box of \{1,2,3,4,5,6\}.

\[
\begin{array}{ccccccc}
  & X & Z & 1 & 2 & 3 & 4 & 5 & 6 \\
 1 & 0 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 \\
 2 & 1/30 & 0 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 \\
 3 & 1/30 & 1/30 & 0 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 \\
 4 & 1/30 & 1/30 & 1/30 & 0 & 1/30 & 1/30 & 1/30 & 1/30 \\
 5 & 1/30 & 1/30 & 1/30 & 1/30 & 0 & 1/30 & 1/30 & 1/30 \\
 6 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 & 0 & 1/30 & 1/30 \\
\end{array}
\]

\[P(X=1, Y=1) = 0 \neq P(X=1) \cdot P(Y=1) = 1/36\]
Sequence of 10 coin tosses

Suppose we denote a head by 1 and a tail by 0. Then a sequence of 10 coin tosses can be represented as a sequence of zeros and ones:

$$\omega = 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1.$$
Ω has $2^{10} = 1024$ elements

| 0000000000 | 1000000000 |
| 0000000001 | 1000000001 |
| 0000000010 | 1000000010 |
| 0000000111 | 1000000111 |
| 0000001000 | 1000001000 |
| 0000001011 | 1000001011 |
| 0000001100 | 1000001100 |
| 0000111111 | 1111111111 |
Sequence of 10 coin tosses

We can define 10 new RV’s over \( \Omega \) in the following way:

Outcome: 0100011011;

\[ X^1 = 0, X^2 = 1, X^3 = 0, \ldots, X^{10} = 1. \]

\( X^i \) represents the outcome of the \( i^{th} \) coin toss.
Sequence of 10 coin tosses

We can show that $X_i$'s are pair-wise independent Bernoulli($\frac{1}{2}$) variables by using the first definition of independence.

$$P(X_i = 1 \mid X_j = 1) = P(X_i = 1 \mid X_j = 0) = P(X_i = 1) = \frac{1}{2};$$

In fact it's enough to look at $\omega^1$ and $\omega^{10}$, by symmetry.
\[ P(X_{10}=1 \mid X^1=0) = \frac{1}{2} = P(X_{10}=1) \]

\[
\begin{align*}
0000000000 & \quad 10000000000 \\
0000000001 & \quad 10000000001 \\
0000000010 & \quad 10000000010 \\
0000000011 & \quad 10000000011 \\
0000000100 & \quad 1000000100 \\
0000000101 & \quad 1000000101 \\
0000000110 & \quad 1000000110 \\
\vdots & \quad \vdots \\
0111111111 & \quad 1111111111
\end{align*}
\]

\( \Omega \) has \( 2^{10} = 1024 \) elements
Joint Distribution of Several Random Variables.

The distribution of \((X_1, X_2, \ldots, X_n)\) is called the joint distribution of \(X_1, X_2, \ldots, X_n\):

\[
P(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n),
\]

satisfying

\[
P(x_1, x_2, \ldots, x_n) \geq 0 \quad \text{and} \quad \sum_{\text{all } (x_1, x_2, \ldots, x_n)} P(x_1, x_2, \ldots, x_n) = 1.
\]
The second definition of independence is easily generalized to the case of several RV’s.

We say $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1=x_1, \ldots, X_n=x_n) = P(X_1=x_1) \cdot \ldots \cdot P(X_n=x_n).$$
Functions of Independent RV’s are Independent

If \( X_1, X_2, \ldots, X_n \) are independent then so are the random variables
\[ Y_1 = f_1(X_1) , \ldots , Y_n = f_n(X_n). \]

For any functions \( f_1, \ldots, f_n \) defined on the range of \( X_1, \ldots, X_n \).
Example:

$X_1, X_2, X_3, X_4, X_5$ are the numbers on 5 dice, and $f(x) = 0$ if $x$ is even and $f(x) = 1$ if $x$ is odd. If we let $Y_i = f(X_i)$, then $Y_1, Y_2, Y_3, Y_4, Y_5$ are independent.

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="https://via.placeholder.com/150" alt="Red Dice" /></td>
<td><img src="https://via.placeholder.com/150" alt="White Dice" /></td>
<td><img src="https://via.placeholder.com/150" alt="Red Dice" /></td>
<td><img src="https://via.placeholder.com/150" alt="White Dice" /></td>
<td><img src="https://via.placeholder.com/150" alt="Red Dice" /></td>
</tr>
<tr>
<td>$Y_1 = 1$</td>
<td>$Y_2 = 0$</td>
<td>$Y_3 = 0$</td>
<td>$Y_4 = 1$</td>
<td>$Y_4 = 0$</td>
</tr>
</tbody>
</table>
Disjoint Blocks of Independent RV’s are Independent

Let’s consider the 5 dice again.

Suppose we group them into random vectors.

\((X_1, X_3)\)  \((X_2, X_4)\)  \((X_5)\)
Disjoint Blocks of Independent RV’s are Independent

\[(X_1, X_3) \quad (X_2, X_4) \quad (X_5)\]

These new random vectors are independent.
Disjoint Blocks of Independent RV’s are Independent

If $X_1, \ldots, X_n$ are independent.

And $B_1, \ldots, B_k$ are disjoint subsets of positive integers such that
$B_1 \cup B_2 \cup \ldots \cup B_k = \{1, 2, \ldots, n\}$.

Then the random vectors
$(X_i : i \in B_1), (X_i : i \in B_2), \ldots, (X_i : i \in B_k)$
are independent.
Disjoint Blocks of Independent RV's are Independent

\[(X_1, X_3) \quad (X_2, X_4) \quad (X_5)\]

In this example

\[B_1 = \{1,3\}; \quad B_2 = \{2,4\}; \quad B_3 = 5.\]
Functions of Disjoint Blocks of Independent RV’s are Independent

If $X_1, X_2, \ldots, X_5$ are independent then so are the random variables

$Y_1 = X_1 + X_3^2; \quad Y_2 = X_2 \cdot X_4; \quad Y_3 = X_5$
Disjoint Blocks of Independent RV’s are Independent

\((X_1, X_3)\)  \hspace{1cm}  \((X_2, X_4)\)  \hspace{1cm}  \((X_5)\)

\[Y_1 = X_1 + X_3^2 = 17\]  \hspace{1cm}  \[Y_1 = X_2 \times X_4 = 10\]  \hspace{1cm}  \[Y_3 = X_5 = 6\]
Suppose each trial can result in m possible categories \( c_1, c_2, \ldots, c_m \) with probabilities \( p_1, p_2, \ldots, p_m \), where \( p_1 + p_2 + \ldots + p_m = 1 \).

Suppose we make a sequence of \( n \) independent trials and let \( N_i \) denote the number of results in the \( i^{th} \) category \( c_i \).
Multinomial Distribution

Then for every m-tuple of non-negative integers \((n_1, n_2, \ldots, n_m)\) with \(n_1 + n_2 + \ldots + n_m = n\)

\[
P(N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m) = \frac{n!}{n_1!n_2!\ldots n_m!} p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}
\]
Suppose we roll a fair die 10 times and record the number of

1, 3, 5 and 'even'.

Question:
What’s the probability of seeing

1, 2, 3, and 4 even numbers?
Using the multinomial distribution:

\[ P(N_1=1, N_3=2, N_5=3, N_{\text{even}}=4) = \frac{10!}{1!2!3!4!} \left( \frac{1}{6} \right)^1 \left( \frac{1}{6} \right)^2 \left( \frac{1}{6} \right)^3 \left( \frac{3}{6} \right)^4 \]

\[ = 0.016878858 \]