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Follows Jim Pitman's book:
Probability
Sections 1.6
Multiplication rule for 3 Events

The Multiplication rule for two events says:
\[ P(AB) = P(A) P(B \mid A) \]

The Multiplication rule extend to 3 Events:
\[ P(ABC) = P(AB)P(C \mid AB) = P(A) P(B \mid A) P(C \mid AB) \]
Multiplication rule for n Events

Similarly, it extends to n events:

\[ P(A_1 A_2 \ldots A_n) = P(A_1 \ldots A_{n-1})P(A_n | A_1 \ldots A_{n-1}) \]

\[ = P(A_1) P(A_2 | A_1) P(A_3 | A_1 A_2) \ldots P(A_n | A_1 \ldots A_{n-1}) \]
We roll two dice. What is the chance that we will roll out Shesh Besh: \[
\begin{array}{c}
\text{for the first time on the n’th roll?}
\end{array}
\]

\[
P = \frac{1}{36} \frac{35}{36} \frac{1}{36} \frac{35}{36} \frac{35}{36} \frac{1}{36} \frac{35}{36} \frac{35}{36} \frac{35}{36} \frac{1}{36}
\]
This is a **Geometric Distribution** with parameter $p=\frac{1}{36}$.
The Geometric distribution

In $\text{Geom}(p)$ distribution the probability of the outcome $n$ for $n=1,2,3...$ is given by:

$$p \ (1-p)^{n-1}$$

Sanity check: is $\sum_{n=1}^{1} p(1-p)^{n-1} = 1$?
The Birthday Problem

If there are \( n \) students in the class, what is the chance that at least two of them have the same birthday?

\[
P(\text{at least 2 have same birthday}) = 1 - P(\text{No coinciding birthdays}).
\]

Let \( B_i \) be the birthday of student number \( i \).

The probability of no coinciding birthdays is:

\[
P(B_2 \notin \{B_1\} \& B_3 \notin \{B_1,B_2\} \& \ldots \& B_n \notin \{B_1,\ldots,B_{n-1}\}).
\]
Use multiplication rule to find
\[ P(B_2 \notin \{B_1\} \land B_3 \notin \{B_1, B_2\} \land \ldots \land B_n \notin \{B_1, \ldots, B_{n-1}\}) \].

\[
1 - \frac{1}{365} \rightarrow \quad B_2 \notin \{B_1\} \\
1 - \frac{2}{365} \rightarrow \quad B_3 \notin \{B_1, B_2\} \\
\ldots \\
1 - \frac{n-1}{365} \rightarrow \quad B_n \notin \{B_1, \ldots, B_{n-1}\} \]
The Birthday Problem

\[ P(\text{at least 2 have same birthday}) = 1 - P(\text{No coinciding birthdays}) = \]
\[ 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\ldots\left(1 - \frac{n-1}{365}\right) \]

Q: How can we compute this for large n?

A: Approximate!
The Birthday Problem

\[ \log(P(\text{No coinciding birthdays})) = \]

\[ = \log((1 - \frac{1}{365})(1 - \frac{2}{365}) \ldots (1 - \frac{n-1}{365})) \]

\[ = \log(1 - \frac{1}{365}) + \log(1 - \frac{2}{365}) + \ldots + \log(1 - \frac{n-1}{365}) \]

\[ \approx - \frac{1}{365} - \frac{2}{365} - \ldots - \frac{n-1}{365} \]

\[ = - \frac{1}{365} \left( \frac{1}{2} n(n - 1) \right) \]
The Birthday Problem

\[ P(\text{No coinciding birthdays}) \approx e^{-\frac{n(n-1)}{2 \times 365}} \]

\[ P(\text{At least 2 have same birthday}) \approx 1 - e^{-\frac{n(n-1)}{2 \times 365}} \]
Probability of no coinciding birthday as a function of $n$.
Independence of 3 events

Recall that $A$ and $B$ are independent if:

$$P(B|A) = P(B|A^c) = P(B);$$

We say that $A$, $B$ and $C$ are independent if:

$$P(C|AB) = P(C|A^cB) = P(C|A^cB^c) = P(C|AB^c) = P(C)$$
Independence of n events

The events $A_1, \ldots, A_n$ are independent if

$$P(A_i \mid B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n) = P(A_i)$$
for $B_i = A_i$ or $A_i^c$

This is equivalent to following multiplication rules:

$$P(B_1 B_2 \ldots B_n) = P(B_1) P(B_2) \ldots P(B_n)$$
for $B_i = A_i$ or $A_i^c$
Independence of n events

**Question:** Consider the events $A_1, \ldots, A_n$.
Suppose that for all $i$ and $j$ the events $A_i$ and $A_j$ are independent.

Does that mean that $A_1, \ldots, A_n$ are all independent?
Pair-wise independence does not imply independence

I pick one of these people at random. If I tell you that it’s a girl, there is an equal chance that she is a blond or a brunet; she has blue or brown eyes. Similarly for a boy.

However, if I tell you that I picked a blond and blue eyed person, it has to be a boy.

So sex, eye color and hair color, for this group, are pair-wise independent, but not independent.