On Random Graph Homomorphism into $\mathbf{Z}$

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SRW and Tree indexed random walks

SRW:

- Random $f : [0, N] \to [-N, N]$ defined by
  - $f(0) = 0$.
  - $f(x+1) = f(x) \pm 1$ with prob. 1/2.

- Equivalent definition: let:

$$\tilde{L_N} = \{f : [0, N] \to [-N, N], f(0) = 0\},$$

  and define: $\forall f \in \tilde{L_N}, \mathbf{P}[f] = \frac{1}{|\tilde{L_N}|}$.

- $\mathbf{E}[|f(N)|] \sim N^{1/2}$, and: $\mathbf{E}[|R(f)|] \sim N^{1/2}$ ($R(f) = \{f(x)\}_{x \in [0,N]}$).

Tree indexed-random walk: Let T_N be the N-level binary tree.

- Random $f : T_N \to [-N, N]$ defined by
  - $f(root) = 0$.
  - $f(x) = f(x') \pm 1$ with prob. 1/2 (where x' is the parent of x).
- Equivalent definition: let:

$$\tilde{T_N} = \{f : T_N \to [-N, N], f(root) = 0\},$$

  and define: $\forall f \in \tilde{T_N}, \mathbf{P}[f] = \frac{1}{|\tilde{T_N}|}$.

- $\mathbf{E}[|f(leaf)|] \sim N^{1/2}$, and: $\mathbf{E}[|R(f)|] \sim N$.

Wired boundary conditions: The measure $\mathbf{P}$ for T_N with wired boundary conditions is not Markovian any-more.

But: If $W_N = T_{N-1}$ with all the leaves of T_{N-1} connected to a vertex v^*, then $\mathbf{P}$ is just a uniform measure on:

$$\tilde{W_N} = \{f : W_N \to [-N, N], f(root) = 0\}.$$

Typical graph-homomorphism: Let G_N be the N-dimensional discrete-cube. We know that any $f : G_N \to \mathbf{Z}$ has conc. of measure properties. Does a typical graph homomorphism has STRONGER conc. properties?

Def: A (G, v_0)-indexed RW (A random (G, v_0) homomorphism) is defined by

- $\tilde{G} = \{f : G \to \mathbf{Z}, f(v_0) = 0\}$
- $\forall f \in \tilde{G}, \mathbf{P}[f] = \frac{1}{|\tilde{G}|}$

Remarks:

- G must be bi-partitie.
- If we look at (G, v_0) and (G, v_1) RW then the map $f(x) \to f(x) - f(v_1)$ is $1-1$ and preserves quantities such as: $|R(f)|$ and $f(w_1) - f(w_2)$.

General Properties:

Prop: For all $v \in G$, and $0 \leq s \leq t$ with $t - s$ even we have: $\mathbf{P}[f(v) = s] \geq \mathbf{P}[f(v) = t]$.

Proof(Reflection principle): We build an injective map:

$$M : \{f : f(v) = t\} \to \{f : f(v) = s\}.$$

For f with $f(v) = t$ define:

$$A_f = \{w \in G : \exists v_0 \to v_1 \ldots \to v_l = w, \forall i, f(v_i) < (t+s)/2\}$$

and

$$M(f)(w) = \begin{cases} f(w) & \text{if } w \in A_f \\ t + s - f(w) & \text{if } w \notin A_f \end{cases}$$

Definition: Let μ_1, μ_2 be measures on S^V (S a set of reals). $\mu_1 \preceq_d \mu_2$ if

- for all increasing $g : S^V \to \mathbf{R}$, $\mathbf{E}_{\mu_1}(g) \leq \mathbf{E}_{\mu_2}(g)$.
- $\exists \mu$ on $S^V \times S^V$ with marginals μ_1, μ_2. s.t: $\mu[(x, y) : x \leq y] = 1$. (The equivalence is Strassen Theorem).

Theorem: For (G, v_0)-indexed RW, and all $v \in G$, the dist. of $|f(v)|$ is dominated by the dist. of $|SRW(d(v_0, v))|$.

Cor: The theorem is equivalent to saying that for all $t > 0$,

$$\mathbf{P}[|f(v)| > t] \leq \mathbf{P}[|SRW(d(v, v_0))| > t].$$

This implies e.g: $\mathbf{E}[|f(v)|] \leq Cd(v, v_0)^{1/2}$ and $\mathbf{E}[f^2(v)] \leq d$.

Proof of the Theorem:

Main Lemma: Let $\tilde{G} = \{f : G \to \mathbf{Z}, f(v_0) = 0\}$ and $\mathbf{P}$ uniform on $\tilde{G}$.

$\tilde{G}' = \{f : G \to \mathbf{Z}, f(v_0) = 0, f(v_1) = f(v_2) = \ldots\ = f(v_k)\}$ where $v_1, \ldots, v_k$ are the neighbors of v; $\mathbf{P}'$ uniform on $\tilde{G}'$.

Set $|P|[f] = P[|g| = f]$ and $|P'|[f] = P'[|g| = f]$. Then $|P| \preceq_d |P'|$.

Pf:

- By Holley's inequality, it is enough to show that for all $v \in G, s \in \mathbf{R}$ and all vectors x:

  $|P|[f(v) \geq s|f(V\backslash\{v\}) = x] \leq |P'|[f(v) \geq s|f(V\backslash\{v\}) = x]$

- The interesting case is when $f(V \setminus \{v\}) = 1$.

- There we get:
$$|P|[f(v) = 0|f(V \setminus \{v\}) = 1] = \frac{2^k(f,v)}{2 + 2^k(f,v)}$$
  where $k(f,v)$ is the number of connected components of $\{v : f(v) > 0\}$ that intersects the neighborhood of v. Similarly:
$$|P'|[f(v) = 0|f(V \setminus \ \{v\}) = 1] = \frac{2^{k'(f,v)}}{2 + 2^{k'(f,v)}}$$
  where k' is the same as k, but we count all components intersecting $v_1, v_2, \ldots, v_k$ as one.

- Since $k'(f,v) \leq k(f,v)$ we have: $|P| \preceq_d |P'|$.

Proof of the Thm: By induction on $d = d(v_0, v)$.

- Denote by G' the graph in which we glue all neighbors of v_0 into a single vertex v_1'.

- By the Main Lemma, we know that $|f(v)|$ is dominated by $|f'(v')|$.

- $|f'(v')| = |\left(f'(v_1') - f'(v_0')\right) + \left(f'(v') - f'(v_1')\right)|$

- By the induction hype. the dist. $|f'(v') - f'(v_1')|$ is dominated by the dist. of $|SRW(d-1)|$.

- $f'(v_1') - f'(v_0')$ is a ± 1 variable ind. of $f'(v') - f'(v_1')$.

- **Lemma:** If X, Y are symmetric R.V taking values in $2\mathbf{Z}$ ($2\mathbf{Z}+1$), and $Z \in \{\pm 1\}$ is independent of X, Y. Then if $|X|$ is dominated by $|Y|$, then $|X+Z|$ is dominated by $|Y+Z|$.

Conjecture: $\sup \mathbf{E}[|R(f)|]$ on all graphs G of n vertices is obtained when G is a path of length $n-1$.

Wired Regular Tree

Reminder: T_k is the k-level binary tree. W_k is T_{k-1} with all the leaves of T_{k-1} connected to an additional vertex v^*.

Prop: Let

$$\begin{aligned} P(k,n) &= |\{f : W_k \to \mathbf{Z}, f(root) = 0, f(v^*) = n\}| \\ &= Z_{n,k}\mathbf{P}[f(v^*) = n] \end{aligned}$$

then for all $k \geq 0, n \geq 0$:

$$P(k, n+2) \leq 2^{-2^n} P(k,n).$$

(and therefore $|f(v^*)|$ is tight as $k \to \infty$).

Proof:

$$P(k,n) = (P(k-1,n-1) + P(k-1,n-1))^2$$

Thm: There are constants c_1, c_2 s.t:

$$\lim_{k\to\infty} \mathbf{P}[c_1 \log k \leq R(f) \leq c_2 \log k] = 1$$

The discrete cube

We are interested in the behavior of $(G_k, 0)$-indexed walks where G_k is the k-dimensional discrete cube.

Conj: For all $t > 0$,

$$\lim_{k\to\infty} \mathbf{P}[|R(f)| > tk] = 0$$

Prop: For all $t > 0$, all k, and all $v \in G_k$,

$$\mathbf{P}[|f(v)| > tk] \le 2e^{-kt^2/4}.$$

Pf: Follows from the estimates for SRW.

Alternative Pf (with worse constants):

- Define $S(x) = S_v(f)(x) = f(x) - f(x \oplus v)$.
- Clearly, $\sum_x S(x) = 0$ and for all i, x, y: $\mathbf{P}[S(x) = i] = \mathbf{P}[S(y) = i]$.
- Now by conc. of measure inq.:

$$\begin{aligned}
\mathbf{P}[|f(v)| > tk] &= \mathbf{P}[|S(0)| > tk] = \frac{1}{2^k}\sum_{x\in G_k} \mathbf{P}[|S(x)| > tk] \\
&= \mathbf{E}\left[\frac{|\{x : |S(x)| > tk\}|}{2^k}\right] \le Ce^{-kt^2/2}
\end{aligned}$$