A JUSTIFICATION OF SOME COMMON LAWS OF MORTALITY

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The objectives of this paper are twofold. The first one is to introduce the modern concepts and definitions of the statistical subject of *life testing* and to tie them in with the corresponding concepts of actuarial science.

The second half of the paper presents a justification of some common laws of mortality (Gompertz's, Makeham's I, Makeham's II), and also gives a very general form of a mortality law that may prove useful.

Let $F(x)$ be a *life cumulative distribution function* for some element under consideration. $F(x)$ is thus the probability that this element fails before reaching age x.

Associated with $F(x)$ is a *life density function*, $f(x) = dF(x)/dx$ and a *failure rate at age x*, $\mu(x) = f(x)/[1 - F(x)]$.

From the equation,

$$[1 - F(x)]\,\mu(x)\,\Delta x = f(x)\,\Delta x\,, \qquad (1)$$

one sees that $\mu(x)$ may be interpreted as the conditional probability of failure in the next interval of length Δx, given that the element has survived to age x. This follows from the fact that $[1 - F(x)]$ is the probability that the element survives to age x and $f(x)\Delta x$ is the probability that the element fails in the next interval of length Δx. *E.g.*, if the element is a life aged x,

$$F(x) = {}_xq_0 = 1 - {}_xp_0 = 1 - l_x/l_0$$

$$f(x) = -\frac{d}{dx}(l_x/l_0)$$

$$\mu(x) = -\frac{1}{l_x}\frac{d}{dx}l_x = \mu\,,$$

the *force of mortality at age x*.

Theorem I:

Consider an element (failure rate $\mu(x)$) made up of n components (failure rates $\mu_1(x), \mu_2(x), \ldots, \mu_n(x)$). Let the times of failure of the components be distributed independently. If it is assumed that the element will fail on the first failure of a component, then

$$\mu(x) = \mu_1(x) + \mu_2(x) + \ldots + \mu_n(x)\,.$$

Proof:

Let the life cumulative distribution functions associated with the components be $F_1(x), F_2(x), \ldots, F_n(x)$.

The element survives to at least age x if and only if all the components survive to at least that age.

The probability that the element fails after reaching age x is $1 - F(x)$, and the probability that all the components survive to age x is

$$[1 - F_1(x)][1 - F_2(x)]\ldots[1 - F_n(x)]\,.$$

Thus

$$1 - F(x) = [1 - F_1(x)][1 - F_2(x)]\ldots[1 - F_n(x)]\,.$$

Thus

$$\log[1 - F(x)] = \log[1 - F_1(x)] + \ldots + \log[1 - F_n(x)]\,.$$

Differentiating each side of this identity,

$$-\frac{f(x)}{1 - F(x)} = -\frac{f_1(x)}{1 - F_1(x)} - \ldots - \frac{f_n(x)}{1 - F_n(x)}\,;$$

i.e.,

$$\mu(x) = \mu_1(x) + \mu_2(x) + \ldots + \mu_n(x)\,, \qquad \text{Q.E.D.}$$

Theorem II:

The form of the cumulative distribution function of the smallest member of a sample of n from a fixed distribution must asymptotically be one of the following forms:

$$1 - \exp[-\exp\{a(x - b)\}] \qquad a > 0 \qquad (2)$$

$$1 - \exp[-a(b - x)^{-c}] \qquad a, c > 0\,; \quad x \le b \qquad (3)$$

$$1 - \exp[-a(x - b)^{c}] \qquad a, b, c > 0\,; \quad x \ge b\,. \qquad (4)$$

(NOTE.—a, b, c are functions of n here; $\exp[\phi(x)]$ represents $e^{\phi(x)}$.)

This result was just derived by Fisher and Tippet [1].* A version of the proof may be found in Kendall [2].

Theorem III:

If an element is composed of many identically distributed and independent components, and if the element fails at the first failure of a component, then its failure rate at age x will be approximately one of the following:

$$Bc^x \qquad B, c > 0\,; \quad 0 \le x \le \infty \qquad (5)$$

$$A/(B - x)^{c+1} \qquad A, B, c > 0\,; \quad x \le B \qquad (6)$$

$$H(x - B)^{c-1} \qquad c, H > 0\,; \quad x \ge B\,. \qquad (7)$$

* Bracketed numbers refer to references at the end of this paper.

Proof:

Assume the element will fail on the first failure of a component. From Theorem II we know that the life cumulative distribution function of the first failure (*i.e.*, smallest life) is of the form (2), (3), or (4).

Using the relation $\mu(x) = f(x)/[1 - F(x)]$ it is easily seen that the failure rate must asymptotically be of the form (5), (6), or (7).

Actuarial Applications

If the human body is considered to be an element made up of many components whose lifetimes are independent and identically distributed, then it follows that the force of mortality is of the form (5), (6), or (7). (Note that (5) is Gompertz's Law.)

The restriction of independence and identical distribution may be relaxed considerably. Watson [3] has shown that as long as components sufficiently far apart are independent the limiting forms (2), (3), (4) still apply. ("Sufficiently far apart" has an appropriate definition.)

The death of a person may be caused by many different means, some natural, some not. By the use of Theorem II a resultant force of mortality may be obtained by considering the makeup of each particular means. *E.g.*, there are components, the chance of whose failure is independent of age. This implies that a constant term should be included in the force of mortality. This constant term will also take into account external forces acting on the body which are reasonably independent of age, *e.g.*, the chance of accidental death.

Conclusions

The most general form of the force of mortality obtainable under the above conditions is of the form:

$$\sum_i H_i (x - B_i)^{c_i - 1} + \sum_j \frac{A_j}{(b_j - x)^{c_j + 1}} + \sum_k E_k d_k^x, \qquad (8)$$

where the summations should be interpreted in the most general way. The parameters are functions of n_i, n_j, n_k respectively and may apply for only certain ranges of age. (Note: taking a $d_k = 1$ provides a constant term.)

Gompertz's, Makeham's First, and Makeham's Second Laws are all contained in (8).

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