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SUBJECT: The Practical Realization of a Hilbert Transformation -

DATE: March 13, 1963

FROM: D. R. Brillinger

MEMORANDUM FOR FILE

1. Introduction and Summary

A concept that has been finding increasing use in the analysis of one dimensional, real stochastic processes is that of an analytic signal, (see 1-4). This concept has proven useful because it allows one to give a mathematically precise definition of the notions of envelope, instantaneous phase, and frequency, that fits in with the heuristic conception that communication engineers have long had of these quantities. The purpose of this memorandum is to develop practical formulas that may be used to estimate the analytic signal, envelope, instantaneous phase and instantaneous frequency corresponding to a given observed stretch of a continuous or discrete time series.

To be more explicit let $X(t)$ be a function in $L_2(-\infty,\infty)$ for example, i.e.

$$\int_{-\infty}^{\infty} \{X(t)\}^2 dt < \infty.$$

The analytic signal corresponding to $X(t)$ is defined by $\Phi(t) = \lim_{\theta\to 0} \Phi(z)$, where $z = t + i\theta$, and

$$\Phi(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{X(\tau)}{\tau - z}\, d\tau,$$

the integral being taken in the Cauchy principal value sense. One can show that $\Phi(z)$ is an analytic function for $\theta > 0$ and that $\Phi(t) = X(t) + iY(t)$ where $X(t)$ and $Y(t)$ are related by,

$$X(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Y(\tau)}{\tau - t} \, d\tau$$

$$Y(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(\tau)}{\tau - t} \, d\tau$$

the integrals being taken in the Cauchy principal value sense once again. $Y(t)$ is called the <u>conjugate signal</u> of $X(t)$, $|\Phi(t)| = \sqrt{\{X(t)\}^2 + \{Y(t)\}^2}$ is called the <u>envelope</u> of $X(t)$, $\arg \Phi(t) = \varphi_t$ is called the <u>instantaneous phase</u> and $\frac{1}{2\pi} \frac{d}{dt} \varphi_t$ the <u>instantaneous frequency</u>.

It can be seen that $Y(t)$ above is the Hilbert transform of $X(t)$, consequently in order to estimate the above defined quantities one needs a practical method of realizing a Hilbert transform given a particular observed signal.

Continuing, suppose that one has observed a stretch of the real time series,

$$\{X_j,\ j = 0, \pm 1, \pm 2, ..\} \text{ if discrete}$$

$$\{X(t),\ -\infty < t < \infty\} \text{ if continuous.}$$

Assume that the series has a harmonic representation of the form,

$$X_j = \int_{-\pi}^{\pi} e^{ij\omega} dz(\omega) \qquad (1) \text{ if discrete}$$

$$X(t) = \int_{-\infty}^{\infty} e^{it\omega} dz(\omega) \qquad (1)* \text{ if continuous.}$$

The conjugate series of the given series in this case are given by,

$$X_j^* = -i \int_{-\pi}^{\pi} (\operatorname{sgn} \omega) e^{ij\omega} dz(\omega) \qquad (2)$$

$$X^*(t) = -i \int_{-\infty}^{\infty} (\operatorname{sgn} \omega) e^{it\omega} dz(\omega) \qquad (2)*$$

Formulae corresponding to (1), (1)*, (2), (2)* demonstrating the real quality of the given series are,

$$X_j = 2 \left[\int_0^{\pi} (\cos j\omega d\alpha(\omega) + \sin j\omega d\beta(\omega)) \right] \qquad (3)$$

$$X_j^* = 2 \left[\int_0^{\pi} (\sin j\omega d\alpha(\omega) - \cos j\omega d\beta(\omega)) \right] \qquad (3)*$$

$$X(t) = 2 \left[\int_0^{\infty} (\cos j\omega d\alpha(\omega) + \sin j\omega d\beta(\omega)) \right] \qquad (4)$$

$$X^*(t) = 2 \left[\int_0^{\infty} (\sin j\omega d\alpha(\omega) - \cos j\omega d\beta(\omega)) \right] \qquad (4)*$$

where

$$z(\omega) = \alpha(\omega) - i\beta(\omega).$$

The problem is now as follows; from a given observed stretch of X_j, $\{X(t)\}$, estimate the corresponding stretch of X_j^*, $\{X^*(t)\}$.

On the way to the development of the computing formulae mentioned above, a method is developed for the practical realization of a filter with any desired transfer function.

2. Filter Design

Consider the following general problem; it is desired to find a finite Fourier approximation to a function with a Fourier series expansion on $[-\pi, \pi]$.

i.e., given

$$f(\omega) = \frac{1}{\pi}\left[\frac{1}{2}a_o + \sum_{k=1}^{\infty} a_k \cos k\omega + b_k \sin k\omega\right] - \pi \leq \omega \leq \pi$$

where

$$a_k = \int_{-\pi}^{\pi} f(\omega) \cos \omega k d\omega, \quad b_k = \int_{-\pi}^{\pi} f(\omega) \sin \omega k d\omega,$$

it is desired to approximate $f(\omega)$ by a finite Fourier series say,

$$g(\omega) = \frac{1}{\pi}\left[\frac{1}{2}\lambda_o a_o + \sum_{k=1}^{n} \lambda_k (a_k \cos\omega\, k + b_k \sin \omega k)\right]$$

Suppose,

$$H(\omega) = \frac{1}{\pi}\left[\frac{1}{2}\lambda_o + \sum_{k=1}^{n} \lambda_k \cos \omega k\right]$$

$$\therefore\ g(\omega) = \int_{-\pi}^{\pi} H(\omega-y)f(y)dy = \int_{-\pi}^{\pi} H(y)f(\omega-y)dy$$

Consequently to effect the desired approximation one sees that it is desirable to have the "window" $H(\omega)$ approximate the Dirac δ-function. Put this way the given problem is equivalent to the problem of the choice of a window through which to view the power spectrum when attempting to estimate it. Solutions which have been proposed for this problem include,

$$\lambda_k = 1;\ 1 - k/n;\ 1 - 2a + 2a \cos{}^{\pi k}/_{n};$$

$$\frac{\sin{}^{k\pi}/_{n}}{k\pi/n}.$$

See [1] for example.

The result obtained is thus the following; if one wishes a finite Fourier approximation one should take the product of the k-th coefficients in the infinite Fourier series and a convergence factor λ_k. A reasonable choice of the λ_k appears to be any choice that works well in the analysis of power spectra.

In the continuous case, suppose,

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}F(t)dt.$$

Approximation is to be made by

$$g(\omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{-i\omega t}\Lambda(t)F(t)dt.$$

Let

$$H(\omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{-i\omega t}\Lambda(t)dt.$$

Then,

$$g(\omega) = \int_{-\infty}^{\infty} H(\omega-y)f(y)dy$$

as before, and the same considerations as given previously indicate reasonable choices of $\Lambda(t)$ as being,

$$\Lambda(t) = 1;\ 1 - t/T;\ 1 - 2a + 2a\cos \pi t/T;$$

$$\frac{\sin \pi t/T}{\pi t/T}.$$

3. Hilbert Transform

Suppose that one wishes to estimate the conjugate series corresponding to a given observed stretch of a time series by means of a linear filter applied to the observed stretch, i.e., estimate,

$$X_j^* \quad \text{by } X_j' = \sum_{-n}^{n} a_k X_{j-k} \tag{3}$$

or

$$X^*(t) \text{ by } X'(t) = \int_{-T}^{T} A(s)X(t-s)ds \tag{4}$$

Consider the discrete case,

From (3)

$$X_j' = \sum_{-n}^{n} a_k \int_{-\pi}^{\pi} e^{i(j-k)\omega} dz(\omega)$$

$$= \int_{-\pi}^{\pi} e^{ij\omega} \left\{ \sum_{-n}^{n} a_k e^{-ik\omega} \right\} dz(\omega). \qquad (5)$$

Comparing this to (2), one wishes

$$\sum_{-n}^{n} a_k e^{-ik\omega} \text{ near } -i \text{ sgn } \omega.$$

Choose $a_{-k} = -a_k$.

$\therefore$ One wishes $2 \sum_{1}^{n} a_k \sin k\omega$ near sgn ω.

The Fourier expansion of sgn ω is

$$\text{sgn } \omega = \frac{4}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right).$$

Considerations of the previous section indicate as a reasonable choice of a_k,

$$a_k = \frac{2}{\pi} \cdot \frac{1}{k} \lambda_k, \quad k \text{ odd}$$

$$= 0 \qquad k \text{ even},$$

the method of choosing λ_k having been indicated previously.

Summing up a reasonable estimate of X_j^* would appear to be,

$$X_j' = \sum_{-n}^{n} a_k \, X_{j-k},$$

$$a_{-k} = -a_k; \; a_{2k} = 0; \; a_{2k+1} = \frac{2}{\pi} \cdot \frac{1}{2k+1} \cdot \lambda_{2k+1}.$$

In the continuous case similar considerations indicate as a choice of $X'(t)$,

$$X'(t) = \frac{1}{\pi} \int_{-T}^{T} \frac{\Lambda(s)}{s} X(t-s) ds.$$

Note that in this case, as $T \to \infty$, a good choice of $\Lambda(s)$ is 1 and

$$X'(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(t-s)}{s} \, ds = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(s)}{t-s} \, ds,$$

i.e., it is the Hilbert transformation of $X(t)$ as desired.

4. General Filter

Suppose that one wishes to realize a filter with transfer function $Y(\omega)$, i.e., given a stretch of time series X_j or $X(t)$ one wishes the corresponding stretch of filtered time series X_j' or $X'(t)$ where

$$X_j' = \int_{-\pi}^{\pi} Y(\omega) \; e^{ij\omega} dz(\omega)$$

$$X'(t) = \int_{-\infty}^{\infty} Y(\omega) \; e^{it\omega} dz(\omega).$$

for some $Y(\omega)$.

As in section 2, to realize the required filter all that one needs to do is to perform a Fourier series expansion and then use a finite part of that series in conjunction with a set of convergence factors λ_k. E.g. suppose one wishes to construct a filter whose effect on a given discrete time series will be to leave the power spectrum unaltered, but to cause a phase advance of α for frequencies between ω_1 and ω_2,

$$\therefore\ Y(\omega) = e^{i\phi(\omega)}$$

where

$$\begin{aligned}\phi(\omega) &= \alpha \text{ for } \omega_1 \le \omega \le \omega_2 \\ &= 0 \text{ for } 0 \le \omega < \omega_1 \\ &\qquad\quad \omega_2 \le \omega \le \pi\end{aligned}$$

and for negative frequencies $Y(\omega)$ is obtained from $Y(-\omega) = Y^*(\omega)$, the condition for a realizable filter.

In this case,

$$\begin{aligned}2\int_0^{\pi} e^{i\omega k} Y(\omega)\,d\omega &= 2\left[\frac{e^{i\omega_1 k}-1}{ik} + \frac{e^{i\pi k}-e^{i\omega_2 k}}{ik}\right] + 2e^{i\alpha}\left[\frac{e^{i\omega_2 k}-e^{i\omega_1 k}}{ik}\right] \qquad & k \neq 0 \\ &= f_k + ig_k & k \neq 0 \\ &= 2\pi & k = 0\end{aligned}$$

To realize this transfer function from (5), $Y(\omega)$ will be approximated by

$$\sum_{-n}^{n} a_k e^{-ik\omega} = a_o + \sum_{1}^{n} (a_k + a_{-k}) \cos k\omega$$

$$+ i \sum_{1}^{n} (a_{-k} - a_k) \sin k\omega.$$

The preceding conditions indicate the choice,

$$a_o = 2\pi\lambda_o$$

$$a_k + a_{-k} = f_k\lambda_k$$

$$a_{-k} - a_k = g_k\lambda_k$$

$$a_{-k} = \frac{f_k + g_k}{2}\lambda_k$$

$$a_k = \frac{f_k - g_k}{2}\lambda_k$$

In general if,

$$Y(\omega) = \frac{1}{2} f_o + \Sigma f_k \cos \omega k + i \Sigma g_k \sin \omega k$$

one takes a_k given by the immediately preceding relations.

5. Estimation of the Envelope, Phase and Frequency

Having estimated the conjugate signal to a given observed signal and consequently the analytic signal, one is now in a position to attempt to estimate the corresponding envelope, phase and frequency.

One might think that he should estimate the envelope by,

$$R(t) = \sqrt{\{X(t)\}^2 + \{X'(t)\}^2}$$

for example. However it is a fact that a signal and its conjugate signal have the same power spectrum, so it is more meaningful to calculate the envelope after the given signal has been filtered by a filter that does not affect phases and that has a similar effect on the power spectrum to the effect of the filter used in the estimation of the conjugate signal.

The instantaneous phase is given by,

$$X(t) = R(t) \cos \varphi_t, \; Y(t) = R(t) \sin \varphi_t.$$

Due to sampling variability φ_t when calculated from the above formulas may jump about considerably, and consequently one may wish to smooth the estimates in some way. A reasonable way to proceed is as follows, smooth the $\cos \varphi_t$ and $\sin \varphi_t$ calculated from the above formulas by some sort of a moving average, call the results α_t and β_t. Estimate the phase by φ_t where

$$\alpha_t + i\beta_t = r_t e^{i\varphi_t}.$$

It may be difficult to estimate the instantaneous frequency, $\frac{1}{2\pi} \frac{d}{dt} \varphi_t$, due to the fact that φ_t skips through multiples of 2π. A formula that avoids this difficulty is,

$$\frac{d}{dt}\varphi_t = \cos \varphi_t \frac{d}{dt} \sin \varphi_t - \sin \varphi_t \frac{d}{dt} \cos \varphi_t.$$

6. Choice of the Convergence Factors

A variety of convergence factors λ_k have appeared in the literature. Calculations were carried out for the choice of n = 7 and the results are appended in graphical form. The function that one is trying to approximate is sgn ω, (equivalent to the search for a Hilbert transform in the discrete case). A list of the λ_k's employed follows with the originator's name.

λ_k	Originator
1.	FOURIER
$1 - K/N$	FEJER
$1 - K^2/N^2$	CESARO
$(\text{SIN}\ \pi K/N)/(\pi K/N)$	RIEMANN
$(1-K/N)\text{COS}\ \pi K/N + \frac{1}{\pi}\ \text{SIN}\ \pi K/N$	BOHMAN
$\begin{cases} 1 - 6K^2(1-K/N)/N^2 & K \leq N/2 \\ 2(1-K/N)^3 & K \geq N/2 \end{cases}$	JENSEN - DE LA VALLE POUSSIN
$.54 + .46\ \text{COS}\ \pi K/N$	TUKEY
$\begin{cases} e^{-K^2/N^2}\ (\text{COS}\ 3\pi K/2N)/(1-9K^2/N^2) & K \neq N/3 \\ \pi/4 & K = N/3 \end{cases}$	PRESENT WRITER

7. Acknowledgement

The writer would like to acknowledge the help of Professor J. W. Tukey especially in connection with the practical suggestions of Section 5.

MH-1215-DRB-mgw

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Att.
Figures 1-8

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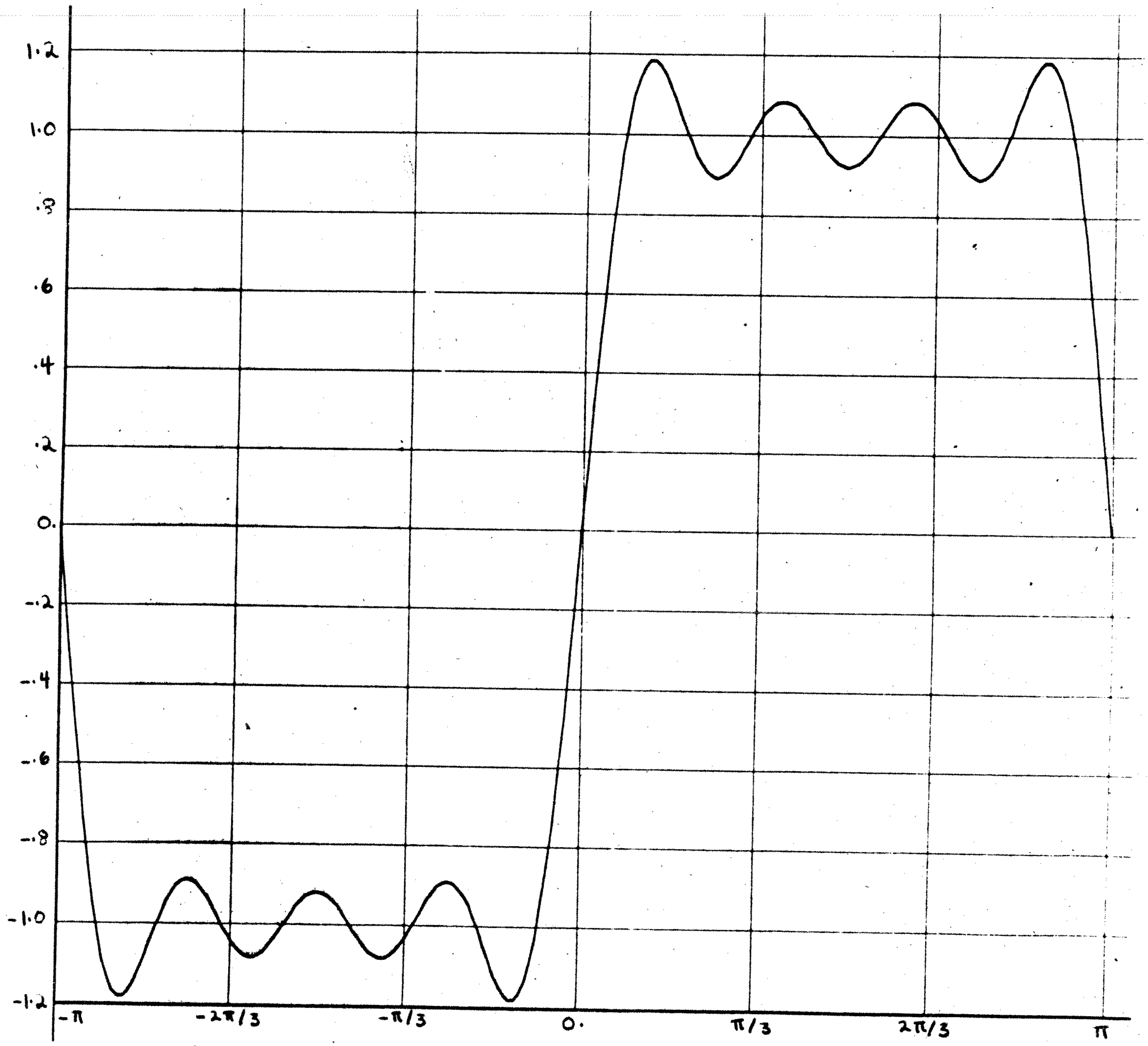
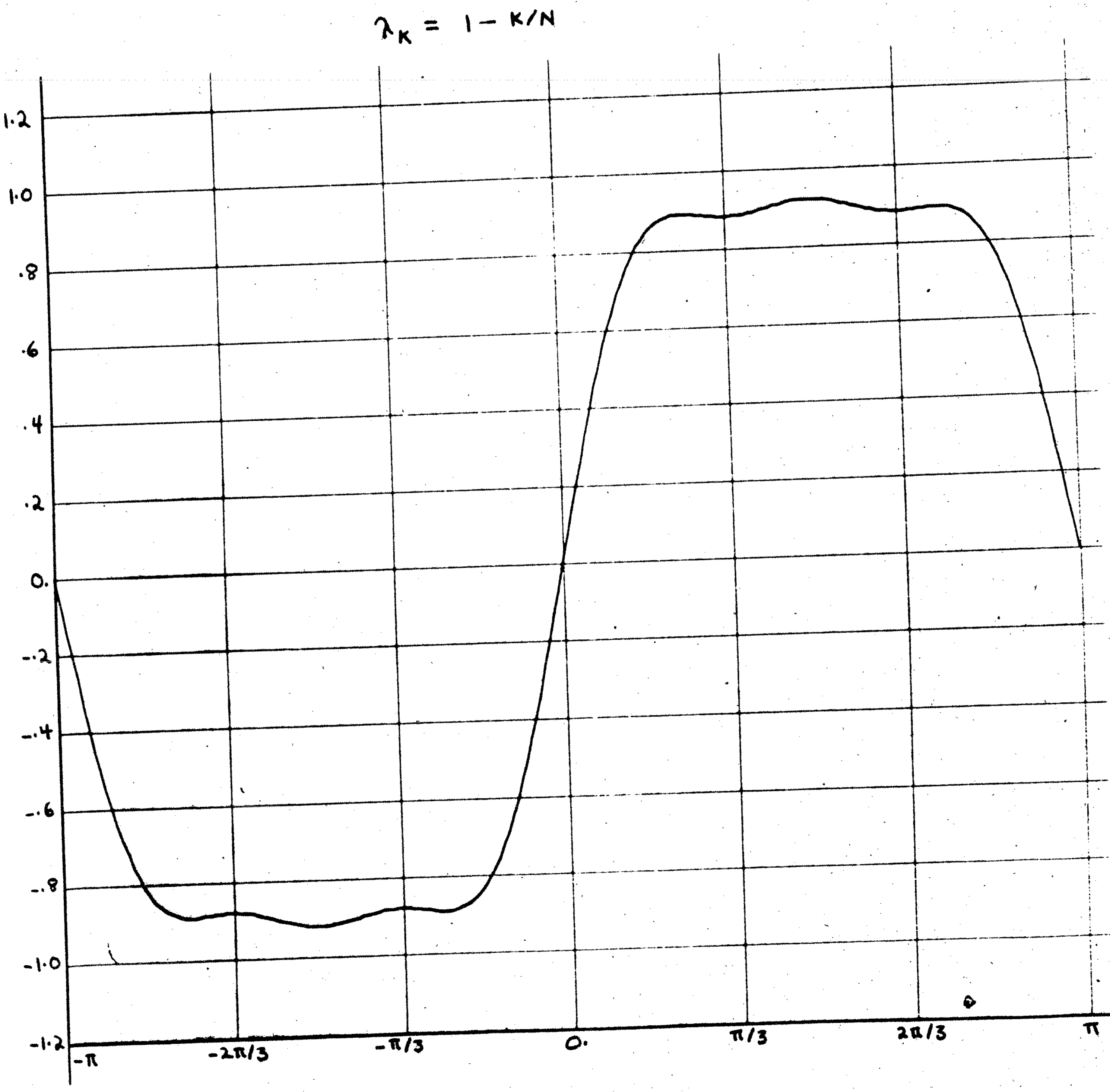
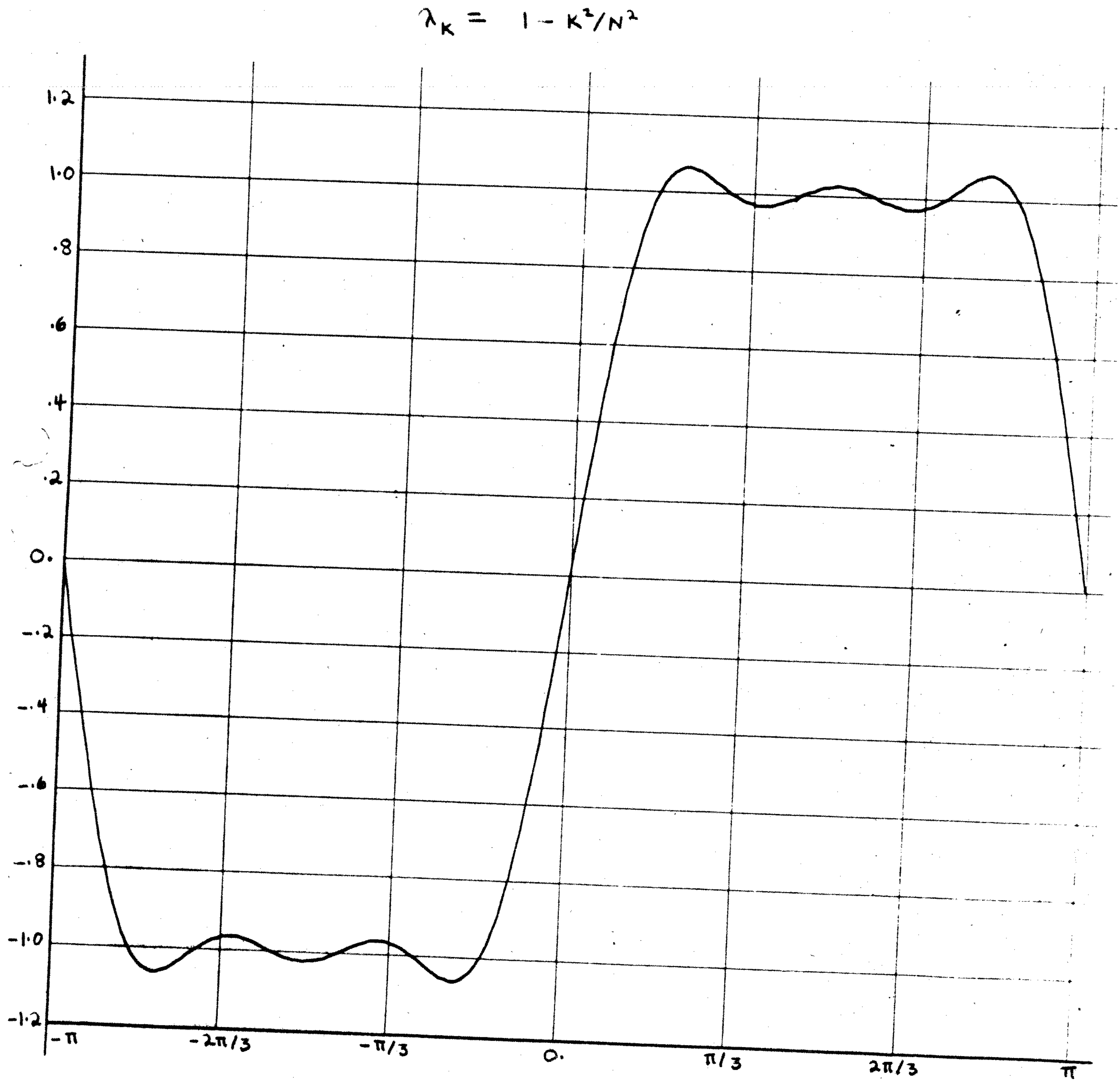
5. G. M. JENKINS, <u>General considerations in the analysis of spectra</u>. Technometrics, 3, pp. 133-166.

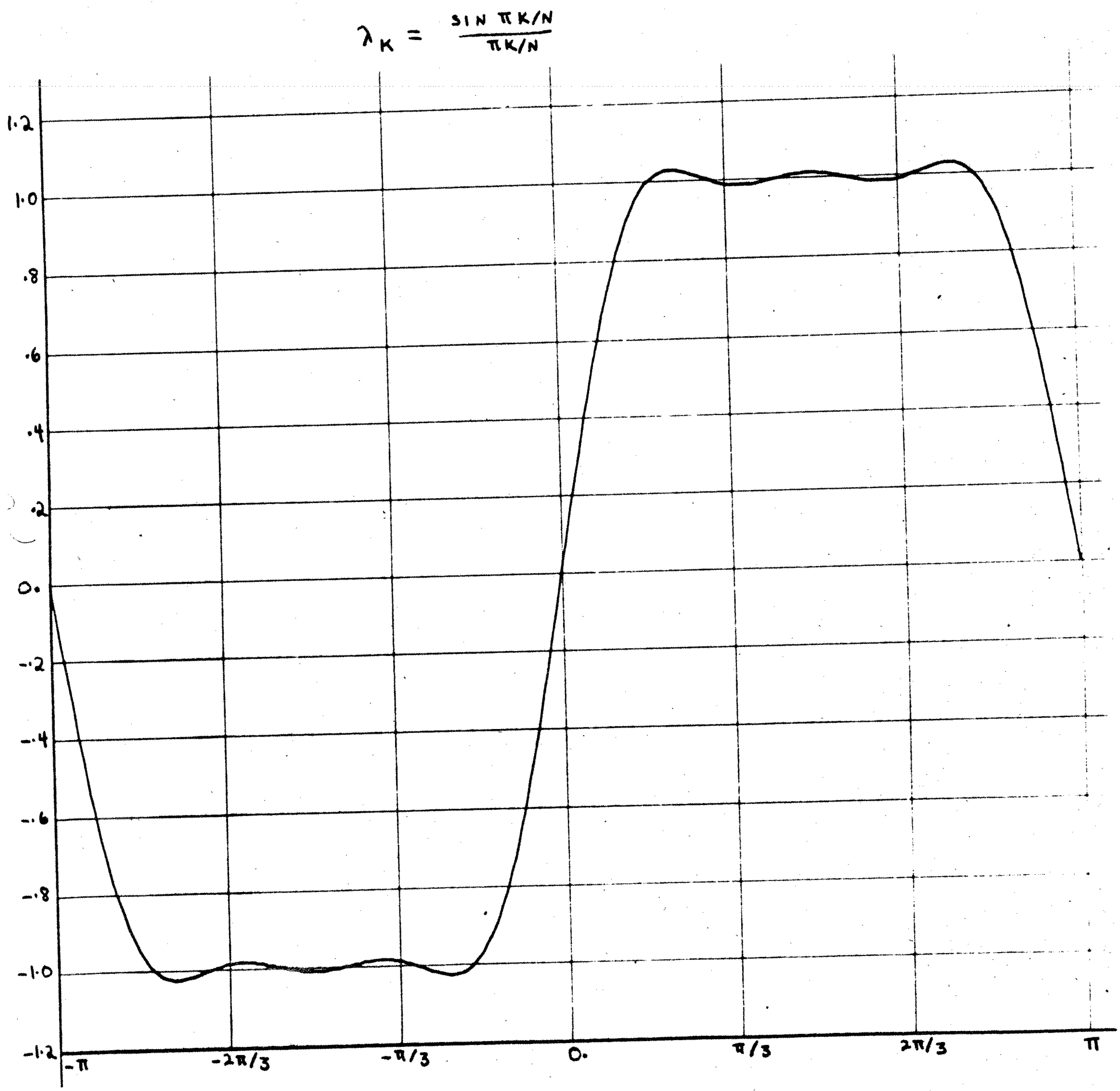
$\lambda_K = 1.$



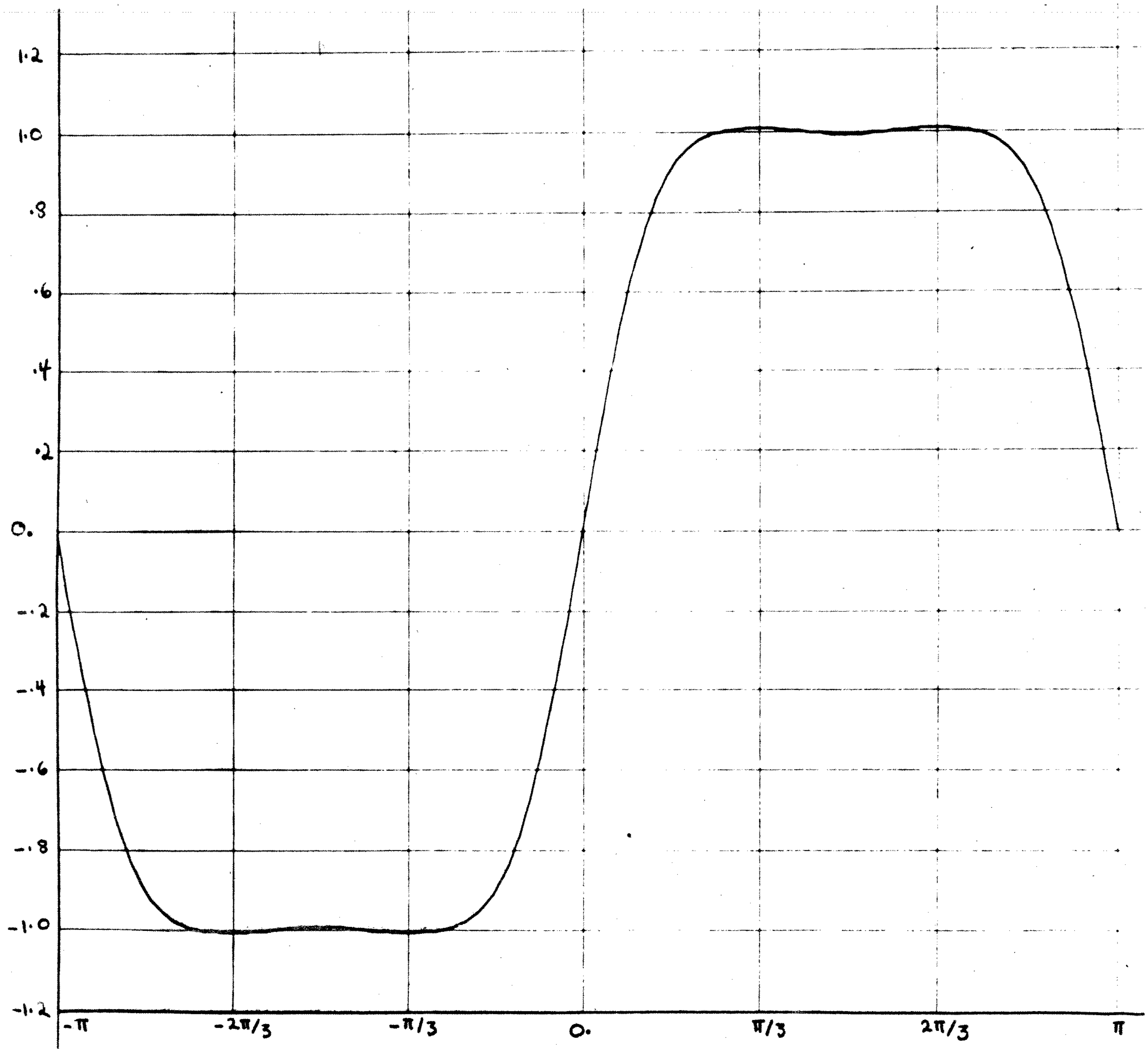


$\lambda_K = 1 - K/N$
1.2
1.0
.8
.6
.4
.2
0.
-.2
-.4
-.6
-.8
-1.0
-1.2
-π
-2π/3
-π/3
0.
π/3
2π/3
π

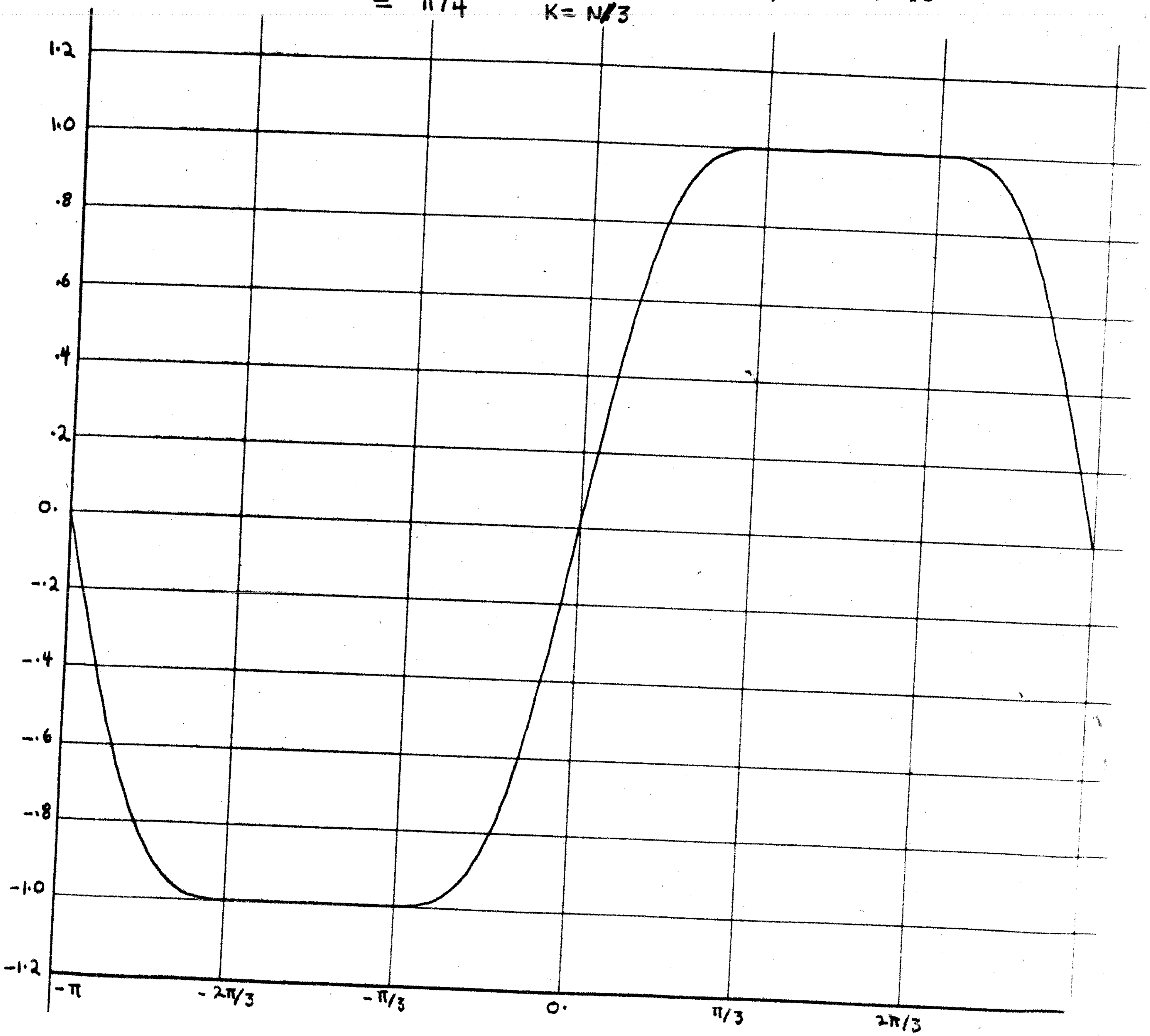

$\lambda_K = 1 - K^2/N^2$
1.2
1.0
.8
.6
.4
.2
0.
-.2
-.4
-.6
-.8
-1.0
-1.2
$-\pi$
$-2\pi/3$
$-\pi/3$
0.
$\pi/3$
$2\pi/3$
π


$\lambda_K = \frac{\sin \pi K/N}{\pi K/N}$
1.2
1.0
.8
.6
.4
.2
0.
-.2
-.4
-.6
-.8
-1.0
-1.2
$-\pi$
$-2\pi/3$
$-\pi/3$
0.
$\pi/3$
$2\pi/3$
π

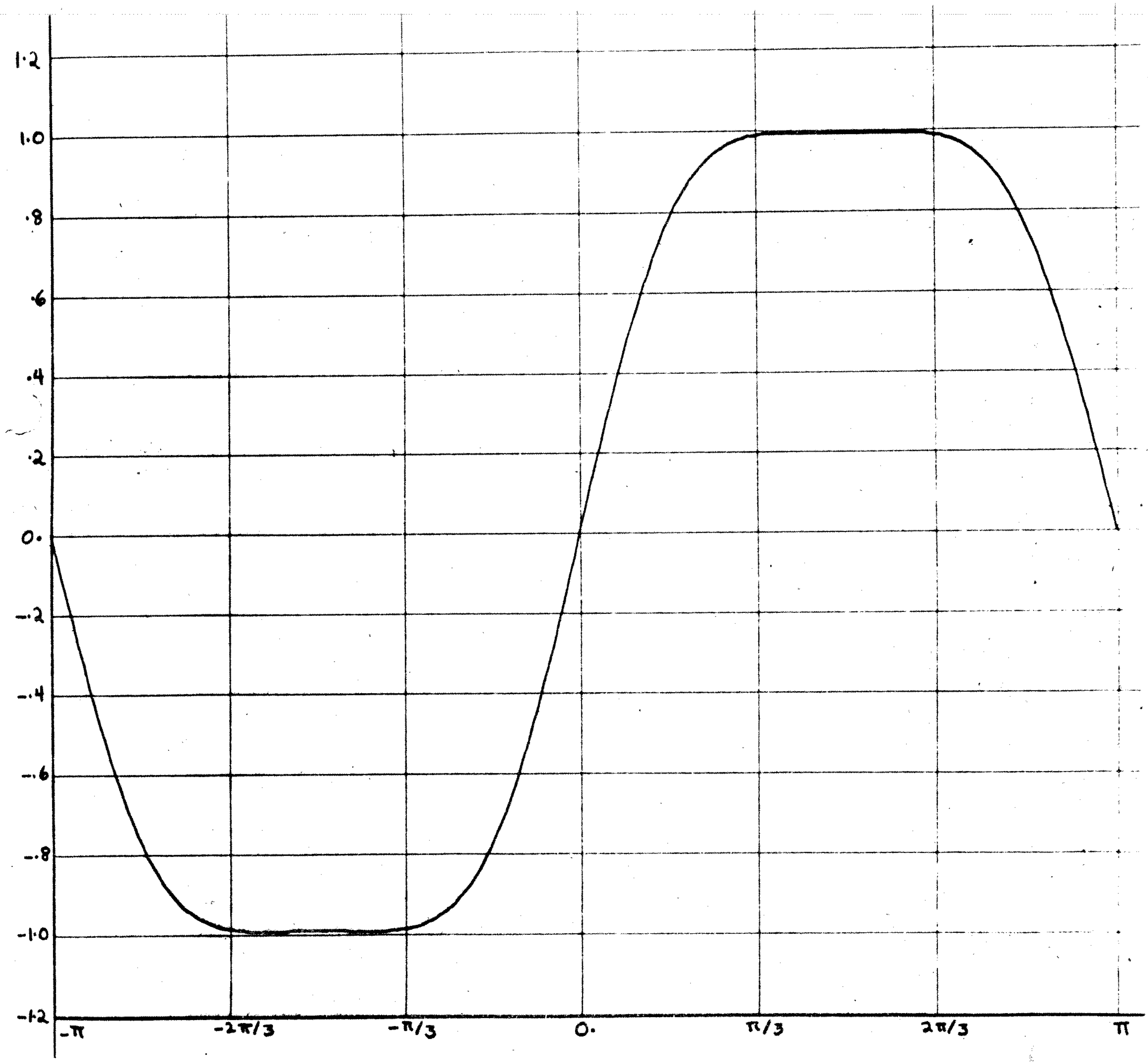
$\lambda_k = .54 + .46 \cos \pi K/N$



$$\lambda_K = e^{-K^2/N^2}(\cos 3\pi K/2N)/(1-9K^2/N^2) \qquad K \neq N/3$$
$$= \pi/4 \qquad K = N/3$$



$$\lambda_K = (1 - K/N)\cos \pi K/N + \frac{1}{\pi}\sin \pi K/N$$



$$\lambda_K = \begin{cases} 1 - 6K^2(1-K/N)/N^2 & 0 \le K \le N/2 \\ 2(1-K/N)^3 & N/2 \le K \le N \end{cases}$$