Space Debris: Flux in a Two Dimensional Orbit

David R. Brillinger*

Abstract

The debris circling the Earth has become an important environmental problem. In particular it creates nonnegligeable risk for spacecraft and satellites. This paper discusses a method to bound the rate of passage of objects through an arc segment in the plane of motion, given random initial conditions for the orbiting object. An advantage of the approach of this paper is that the initial conditions may be arbitrary and that moments beyond the first may be computed in a similar manner.

1 Introduction

There is now a substantial amount of debris in orbit about the Earth. It poses a risk to satellites and space craft and other debris. Unintended breakups as well as collisions occur. The objects in orbit include meteorites, solar system probes, functional spacecraft and debris made up of nonfunctional space craft, fragmentation debris and mission-related debris rocket bodies, [6, 8, 9]. Debilitating collisions have occurred, eg. an Ariane piece collided with a Cérise spacecraft, see [8]. Figure 1 is a NASA graphic illustrating some of the objects in low earth orbit at one point in time.

It is important to understand the risk associated with such debris. This paper is concerned with developing bounds for the intensity of passage of an object orbiting in a plane, through a short curve segment (or arc) in that

---

*Statistics Department, University of California, Berkeley, California, 94720-3860, USA
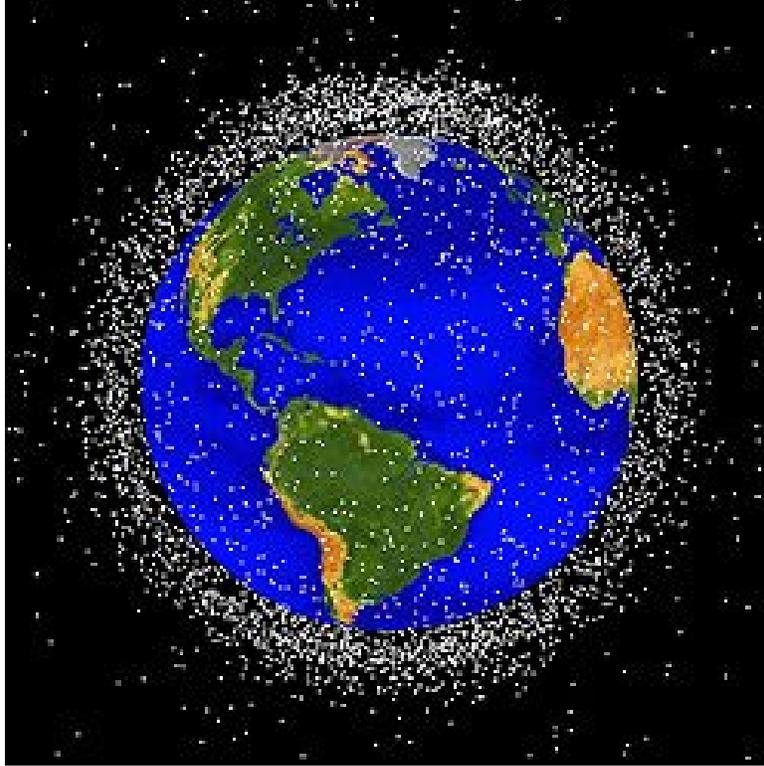
Figure 1: Debris in low earth orbit. www address: sn-callisto.jsc.nasa.gov

plane. The curve segments may be put together to form the boundary of an arbitrary shape and expressions obtained for risks of general objects.

This present paper refers to work in progress and in keeping with that situation is heuristic in approach.

2   Space hazard

Space debris risk assessment may be defined as: the estimation of the probability that certain performance variates, of a space object of interest, exceed relevant critical levels, within a specified time period, as a result of collisions with space debris. For example it may be of interest to estimate

$$Prob\{debris\ passes\ through(\mathbf{x}, \mathbf{x} + d\mathbf{x}]\ in\ time\ interval\ [0, T]\}$$

where $(\mathbf{x}, \mathbf{x} + d\mathbf{x}]$ refers to a small cell. It is clear that the time period of exposure is crucial to the description and estimation of such risks. The sizes of the debris pieces and of the object are also important in practice. In the present work $\mathbf{x}$ will be taken to lie in the orbital plane of the object.

3   Risk probabilities

A statistical concept that proves exceedingly useful in addressing problems of space risk is that of a *point process*, [4]. Realizations of such a process have the form $\{\mathbf{x}_j : j = 1, 2, ...\}$ with the $\mathbf{x}_j$ distinct points in a Euclidian space. In risk analyses the $\mathbf{x}_j$ refer to the locations of occurrence of some damaging event. Such a point process may also be described by a counting measure, with $N(A)$ = the number of points in a given set A. It is often convenient to work in terms of differentials $dN(A) = N(dA)$ where, for small $|dA|$, $dN(A)$ is 1 or 0 depending on whether or not there is a damaging event in the cell dA.

   In risk analysis the probability of an event occuring in a prespecified set, A, is of particular importance. Under regularity conditions one can write

$$Prob\{N(A) > 0\} = 1 - \sum_k \frac{(-1)^k}{k!} \int_A ... \int_A p_k(\mathbf{x}_1, ..., \mathbf{x}_k)d\mathbf{x}_1...d\mathbf{x}_k \quad (1)$$

where the $p_k(\mathbf{x}_1, ..., \mathbf{x}_k)$ are the k-th order product densities of the process, [4]. Product densities are defined as

$$p_k(\mathbf{x}_1, ..., \mathbf{x}_k)d\mathbf{x}_1...d\mathbf{x}_k = E\{dN(\mathbf{x}_1)...dN(\mathbf{x}_k)\} \; for \; the \; \mathbf{x}_j \; distinct$$

and $j = 1, ..., k$. The expression (1) may be used to develop general bounds for risk, for example

$$\int_A p_1(\mathbf{x})d\mathbf{x} \geq Prob\{N(A) > 0\} \geq \int_A p_1(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_A \int_A p_2(\mathbf{x}_1, \mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2$$

so the risk problem becomes one of approximating product densities, [5].

   In the case of a Poisson process, with intensity function $p_1(\mathbf{x}) = p(\mathbf{x})$, one has the exact expression

$$Prob\{N(A) > 0\} = 1 - exp\{-\int_A p(\mathbf{x})d\mathbf{x}\} \quad (2)$$

The Poisson is often pertinent to work with in risk problems because it arises as a limit when many independent point processes are superposed or when the points are rare, [4].

4   Specifying orbits.

Along the way to obtaining expressions for collision probabilities one needs a method to specify orbits. As Kepler and Newton showed these orbits are ellipses with the Earth at a focus, see Figure 2. In this paper the work will be for orbits within a given plane.

There are a number of coordinate systems. Refering to Figure 2 the following (partly classical) notation will be used:

F: a focus
FA: the base line
O: the center of the ellipse
B: the perigee, i.e. point on the orbit closest to F
a: the length of semimajor axis OB. $a > 0$
e: the eccentricity of the ellipse. $0 \leq e < 1$
ω: the argument of perigee, i.e. the angle AFB. $0 \leq \omega < 2\pi$
f: the true anomaly, i.e. the angle BFC. $0 \leq f < 2\pi$
T: time of a perigee passage
C: the location of the object on its orbit at a given time, t

The elements, a, e, give the size and shape of the ellipse. The angles ω and f are measured in the direction of motion of the object around the orbit. To fully specify the movement that direction also needs to be specified. The angle ω gives the orientation of the major axis relative to the base line FA.

Let P denote the object's period. Suppose that $0 \leq T < P$, i.e. T is the time of first perigee passage since time 0. In this work it will be assumed that the initial condition for (ω, T) are stochastic eg. ω is uniform on $[0, 2\pi)$ and independently T is uniform on $[0, P)$. Figure 3 provides a graph of some realizations of orbits for this process. The eccentricity is .9 .

The following coordinate system is particularly convenient for describing a particular location in the plane of motion:

4
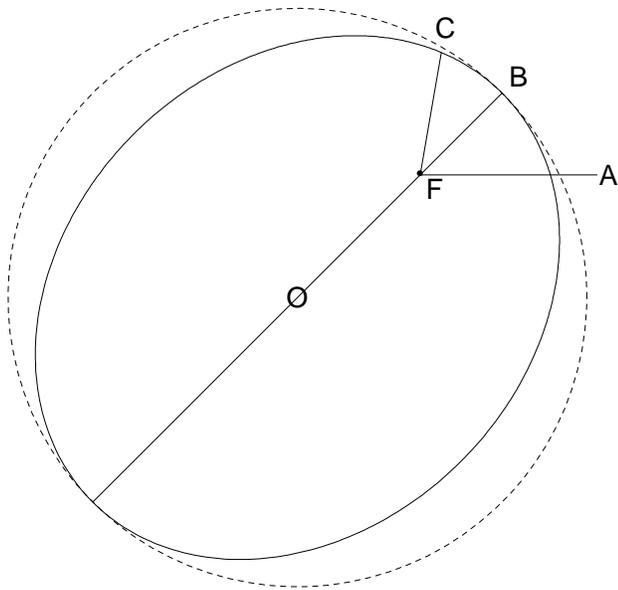
Figure 2: An orbit with eccentricity .6 . The circumscribing circle is given as the curve of dashes.
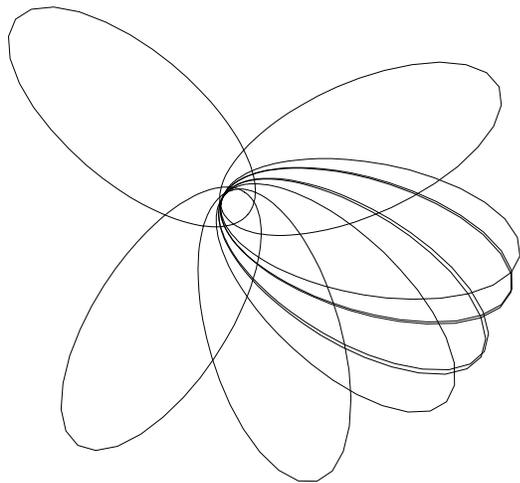
Figure 3: Some realizations of orbits for random initial conditions.

r : radial distance from the focus F to the location C on the ellipse. $r > 0$

ϕ: the angle AFC . $0 \leq \phi < 2\pi$

The value of r is limited, $q < r < q'$ where $q = a(1-e)$, and $q' = a(1+e)$. At

time t the object is at position C, (ϕ_t, r_t), with the angle AFB, $\phi_t = \omega + f_t$.

The equations of motion are classic, [7, 11]. Suppose one is concerned with the location of the object on its orbit at time t. Let E_t, $0 \leq E_t < 2\pi$ be defined via

$$cos\ f_t = (cos\ E_t - e)/(1 - e\ cos\ E_t),\ \ sin\ f_t = \sqrt{1 - e^2} sin\ E_t\ /(1 - e\ cos\ E_t) \tag{3}$$

This E_t, is called the eccentric anomaly and is the angle BOD from the axis OB perpendicularly up to the point D on the circumscribing circle, passing through C. The circle is given in the figure.

As time t passes and E_t goes from 0 to 2π the the ellipse is swept out via

$$r_t = a(1 - e\ cos\ E_t) \tag{4}$$

In particular one has Kepler's equation relating E_t to time t

$$n(t - T) = E_t - e\ sin\ E_t \tag{5}$$

with n the so-called mean motion. This equation has a unique solution for E_t given the lefthand side, p. 192 in [2]. Equation (5) shows that the period of the object is given by $P = 2\pi/n$.

5   Rate computation

This work starts with the case of a single object orbiting in the plane. Its orbit may be denoted (ϕ_t, r_t), $t \geq T$ where ϕ_t is the angle AFC and r_t is the distance FC. A probability of interest concerns the temporal rate of passage through a small arc, eg. one given parametrically by

$$\{(\phi = a(u), r = b(u)),\ u \in U\} \tag{6}$$

where say $U = [0, 1]$ or $= (u, u + \Delta u)$. The arc might also be given as a graph, as in $u = \phi$, $r = b(\phi)$ or as in $\phi = a(r)$ for some range of ϕ or r

values. In a practical case, a satellite or space station might be described by a simply-connected curve.

The locations and times of passages through (6) lead to a two dimensional point process, (t_j, u_j), $j = 1, 2, ...$ with $a(u_j) = \phi_{t_j}, b(u_j) = r_j$. The point (ϕ_{t_j}, r_{t_j}) is the location of the j-th passage through the arc and t_j the time of that passage. If $N(t, u) = \#\{(t_j, u_j) \leq (t, u)\}$ denotes the point process, then the rate function

$$\mu(t, u) = E\{dN(t, u)\}/dt du \tag{7}$$

is of interest. Given μ one can for example compute the expected flux through the arc (6) at time t as

$$\int_U \mu(t, u) du \; / \; \int_U \sqrt{b'(u)^2 + b(u)^2 a'(u)^2} du \tag{8}$$

the denominator being the length of the arc.

The passages through the arc correspond to the solutions (t, u) of the function pair $a(u) = \phi_t$, $b(u) = r_t$. There exists a convenient symbolic representation for the number of such solutions. First consider a real-valued differentiable function, $h(x)$, of a real variable, x, in an interval U. Its number of zeros there may be represented

$$N(U) = \int_U \delta(h(x))|h'(x)|dx$$

where $\delta(.)$ is the dirac delta function, [10]. For m-vector-valued functions, $\mathbf{h}$, of an m-valued $\mathbf{x}$ and a region A one has the analagous expression

$$N(A) = \int_A \delta(\mathbf{h}(\mathbf{x}))|J\mathbf{h}(\mathbf{x})|d\mathbf{x} \tag{9}$$

for the number of solutions of $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ in A. This assumes $\mathbf{h}$ is Lipschitz and writes $J\mathbf{h}$ for the Jacobian, [3].

To use (9) in the present context, take the 2 equations, $a(u) = \phi_t, b(u) = r_t$, in the 2 unknowns t, u, Supposing that the functions a, b are differentiable, the Jacobian, $J\mathbf{h}$, is equal to

$$a'(u)\dot{r}_t - b'(u)\dot{\phi}_t$$

having written $\dot{\phi}$, $\dot{r}$ for $d\phi_t/dt$, dr_t/dt and a', b' for da/du, db/du.

8

Taking A in (9) to be $(t, t + dt) \times U$ one has

$$N(dt, U) = \left(\int_U \delta(\phi_t - a(u)) \delta(r_t - b(u)) |a'(u)\dot{r}_t - b'(u)\dot{\phi}_t| \, du \right) dt \quad (10)$$

Suppose that the variate (ω, T) has density function $p(\omega, T)$ with $0 \leq \omega < 2\pi$ and $0 \leq T < P$. Then the expected value of (10) is

$$\int_E \int_U p \left(a(u) - atan(\sqrt{1 - e^2} sinE, cosE - e), t - \frac{1}{n}(E - esinE) \right) \delta \left(a(1 - cosE) - b(u) \right)$$

$$\cdot |a'(u)aesinE - b'(u)\frac{\sqrt{1 - e^2}}{1 - ecosE}| \, du \, dE \, dt \quad (11)$$

with $q < b(u) < q'$. Details of the derivations and some simplifications are given in the Appendix.

The following examples are useful supposing that ω, T are purely random, i.e. are respectively independent uniforms, $U(0, 2\pi)$, $U(0, P)$. Their joint density $p(\omega, T) = 1/(2\pi P)$.

Example 1. Suppose that the arc of interest is $u = \phi$, $a(u) = \phi$, $b(u) = r$ so $a'(u) = 1$, $b'(u) = 0$, i.e. one has an arc of the circle centered at F of (constant) radius r. The angle ϕ is AFC. The distance r is FC. An example is graphed in Figure 4. From expression (18) in the Appendix, taking U to be $(\phi, \phi + \Delta\phi)$, the desired expected value is

$$\frac{\Delta\phi}{\pi P} \quad (12)$$

provided $q < r < q'$. This does not depend on t or ϕ. If one is concerned with the whole circle the rate is $2/P$, i.e. there are two crosses by the object per orbit.

Example 2. As a second example suppose that the segment of interest is $u = r$, $a(u) = \phi$, $b(u) = r$ so $a'(u) = 0$, $b'(u) = 1$. This is an interval on the radius originating at F at angle ϕ, see Figure 4. Taking U to be $(r, r + \Delta r)$ the expected value, may be evaluated from expression (18) in the Appendix. It is found to be

$$\frac{a\sqrt{1 - e^2}}{r} \frac{1}{\sqrt{(r - q)(q' - r)}} \frac{\Delta r}{\pi P} \quad (13)$$

9
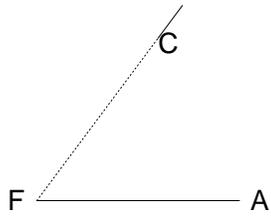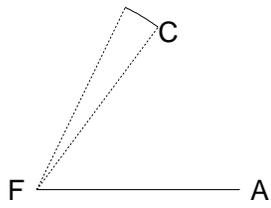
Figure 4: The arcs of Examples 1 and 2.
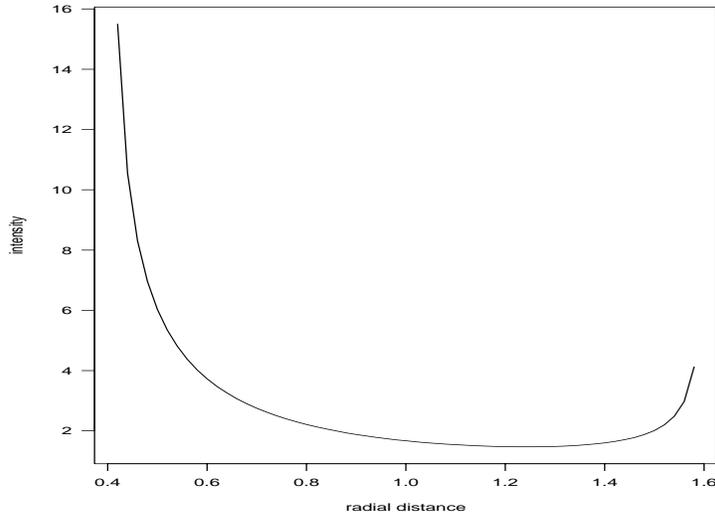
Figure 5: The rate function (13).

where q, $q' = a(1-e)$, $a(1+e)$, $q < r < q'$, $0 \le \phi < 2\pi$. This quantity does not depend on t as could have been anticipated. It is graphed in Figure 5 for the case of eccentricity .6 . The function depends strongly on r. One sees high intensity at the extremes, q, q', of the orbit with the highest at q.

If one inegrates this expression from q to q' one obtains $1/P$ consistent with the object crossing the line segment once per orbit.

6 Discussion

In practice there may be a number of objects orbiting simultaneously. Supposing their initial conditions are sampled independently from $p(\omega, T)$, the rate function remains proportional to that given by expression (11) and for a large number the overall process that is approximately Poisson, so one can obtain approximate risk probabilities as at (2).

Dependence may be introduced amongst particles by assuming a joint distribution for (ω_1, T_1) and (ω_2, T_2). This is necessary, for example, to handle particles resulting from a breakup.

For a product density of order 2 one replaces expression (10) with

$$N(dt_1, U)N(dt_2, U), \ t_1 \ \neq \ t_2$$

Its expected value may be evaluated in the manner of the one dimensional case. Examples will be provided in a later paper.

That the time origin is purely random, i.e. that T is $U[0, P)$, seems plausible for many situations. But it does seem worth assuming that the distribution of ω is not uniform, for example a launch may be made with a prespecified ω_0. Formulae (11) and (18) give results for ω having density $p(\omega)$.

Only the planar case has been considered in this paper. This is what is usually done to begin in the texts on orbital mechanics. The three dimensional case will be addressed in the next paper.

All the assumptions needed for the development of the results have not been presented, eg. conditions allowing exchange of orders of integration. One may avoid the use of th Dirac δ function by formal limiting arguments, as in [3]

7 Acknowledgements

References

[1] Barton, D.K., Brillinger, D. R., El-Shaarawi, A. H., McDaniel, P., Pollock, K. H. and Tuley, M. T. (1998). *Final Report of the Haystack Orbital Debris Data Review Panel.* Technical Memorandum 4809 NASA, Johnson Space Center, Houston.

[2] Battin, R. H. (1987). *An Introduction to the Mathematics and Methods of Astrodynamics.* American Institute of Aeronautics and Astronautics,

New York.

[3] Brillinger, D. R. (1972). On the number of solutions of systems of random equations. *Ann. Math. Statist.* **43**, 534-540.

[4] Brillinger, D. R. (1978). Comparative aspects of the study of ordinary point processes and of time series. pp. 33-133 in *Developments in Statistics* **1**. Academic, New York.

[5] Brillinger, D. R. (1982). Some bounds for seismic risk. Bull. *Bull. Seismol. Soc. America* **72**, 1403-1410.

[6] Cleghorn, G. et al. (1995). *Orbital Debris: a Technical Assessment.* National Academy Press, Washington.

[7] Geyling, F. T. and Westerman, H. R. (1971). *Introduction to Orbital Mechanics.* Addison-Wesley, Reading.

[8] Johnson, N. L. (1998). Monitoring and controlling debris in space. *Scientific American*, August, 62-67.

[9] Johnson, N. L. and McKnight, D. S. (1987). *Artificial Space Debris.* Orbit Book Co., Malabar, Florida.

[10] Kac, M. (1943). On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* **49**, 314-320.

[11] Szebehely, V. G. (1989). *Adventures in Celestial Mechanics.* U. of Texas Press, Austin.

Appendix

In the next develpments it is convenient to work in terms of the angle E_t, indeed [11], p. 48 speaks of "the power of using the 'proper' variables in celestial mechanics".

Following expression (10) one wishes the expected value of

$$N(dt, U) =$$

$$\left(\int_U \delta(\phi_t - a(u))\delta(r_t - b(u))|a'(u)\dot{r}_t - b'(u)\dot{\phi}_t|du \right) dt \qquad (14)$$

with the expected value taken over $(\phi_t, r_t, \dot{\phi}_t, \dot{r}_t)$. These variates need to be expressed in terms of (ω, T) since that is the source of the randomness.

One has equation (4)

$$r_t = a(1 - e\cos E_t)$$

and from it

$$\dot{r}_t = nae\ \sin E_t/(1 - e\cos E_t)$$

since $\dot{E}_t = n/(1 - e\cos E_t)$, [11]. Next, as $\phi_t = \omega + f_t$, from (3)

$$\phi_t = \omega + atan(\sqrt{1 - e^2}\sin E_t\ ,\ \cos E_t\ -\ e)$$

From this last

$$\dot{\phi}_t = n\sqrt{1 - e^2}/(1 - e\ \cos E_t)^2$$

In these terms (14) becomes

$$\left(\int_U \delta\left(\omega\ +\ atan(\sqrt{1 - e^2}\sin E_t\ ,\ \cos E_t\ -\ e)\ -\ a(u) \right) \delta\left(a(1\ -\ e\cos E_t)\ -\ b(u) \right) \right.$$

$$\left. \cdot |a'(u)\,nae\ \sin E_t/(1 - e\cos E_t) - b'(u)n\sqrt{1 - e^2}/(1 - e\cos E_t)^2|\,du \right) dt \quad (15)$$

To continue, E_t as a random variable, needs to be related to (ω, T). From (5) one has

$$-n\ dT = (1\ -\ e\cos E_t)\ dE_t$$

So the probability element

$$p(\omega, T)\ d\omega\ dT$$

$0 \le \omega < 2\pi$, $0 \le T < P$ becomes

$$p(\omega, t\ -\ \frac{1}{n}(E\ -\ e\sin E))\frac{1}{n}(1\ -\ e\cos E)\ d\omega\ dE$$

14

for $0 \leq E < 2\pi$.

For the moment it is convenient to write

$$p_t(\omega, E) = p(\omega, t - \frac{1}{n}(E - e sin E))$$

Using the above expressions, the expected value of (15) may be written

$$\int_E \int_U \frac{1}{n}(1 - e cos E)\, p_t\left(a(u) - atan(\sqrt{1 - e^2} sin E, cos E - e), E)\right)\delta\left(a(1 - cos E) - b(u)\right)$$

$$\cdot |a'(u)nae\frac{sin E}{1 - e cos E} - b'(u)n\frac{\sqrt{1 - e^2}}{(1 - e cos E)^2}|\, du\, dE\, dt \qquad (16)$$

At this point it makes sense to break the integral over E into the intervals $[0, \pi)$ and $[\pi, 2\pi)$ and to use the transformation

$$cos E = \frac{1}{e}\left(1 - \frac{v}{a}\right), \quad sin E = \pm \frac{1}{ea}\sqrt{(v - q)(q' - v)}$$

with the "+" sign for $0 \leq E < \pi$, and the "-" for $\pi \leq E < 2\pi$. In either case

$$dE = \frac{1}{\sqrt{(q' - v)(v - q)}}dv$$

Now expression (?) is

$$\int_q^{q'}\int_U \frac{v}{an}p_t(a(u) - atan(\sqrt{1 - e^2}\frac{1}{ea}\sqrt{(v - q)(q' - v)}, \frac{1}{e}(1-\frac{v}{a})), acos(\frac{1}{e}(1-\frac{v}{a}))$$

$$\cdot\delta(v - b(u))\, |a'(u)nae\frac{1}{ea}\sqrt{(v - q)(q' - v)}\frac{a}{v} - b'(u)\frac{n\sqrt{1 - e^2}}{(v/a)^2}|\sqrt{\frac{1}{(q' - v)(v - q)}}\, dv\, du\, dt$$

$$+$$

$$\int_q^{q'}\int_U \frac{v}{an}p_t(a(u) - atan(-\sqrt{1 - e^2}\frac{1}{ea}\sqrt{(v - q)(q' - v)}, \frac{1}{e}(1-\frac{v}{a})), acos(\frac{1}{e}(1-\frac{v}{a}))$$

$$\cdot\delta(v - b(u))\, |-a'(u)nae\frac{1}{ea}\sqrt{(v - q)(q' - v)}\frac{a}{v} - b'(u)\frac{n\sqrt{1 - e^2}}{(v/a)^2}|\sqrt{\frac{1}{(q' - v)(v - q)}}\, dv\, du\, dt$$

$$(17)$$

In the case that $U = (u, u + \Delta u)$ and that $p(\omega, T)$, is constant, at $1/2\pi P$, this becomes

$$(|a'(u) - b'(u)\frac{a}{b(u)}\frac{\sqrt{1 - e^2}}{\sqrt{(q' - b(u))(b(u) - q)}}$$

$$+ |-a'(u) - b'(u)\frac{a}{b(u)}\frac{\sqrt{1 - e^2}}{\sqrt{(q' - b(u))(b(u) - q)}}|)\frac{\Delta u \ dt}{2\pi P} \qquad (18)$$

provided $q < b(u) < q'$.