Waves in a Spatial Queue: Stop-and-Go at Airport Security

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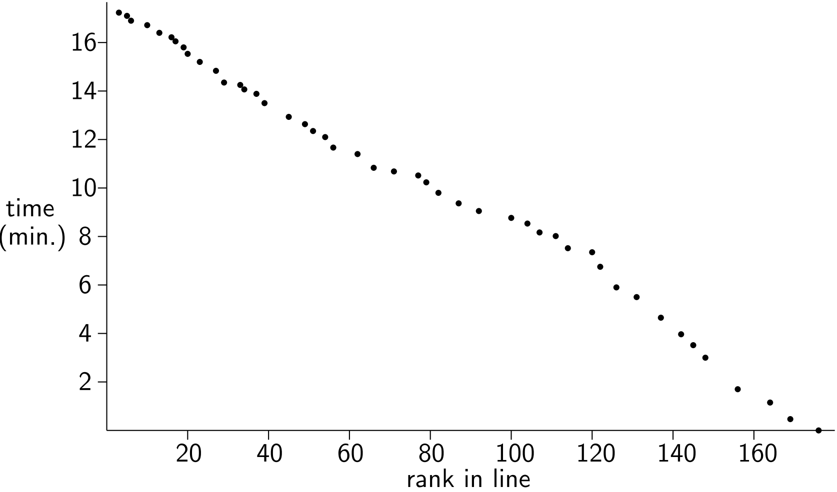
17 minutes in line at security at Oakland airport



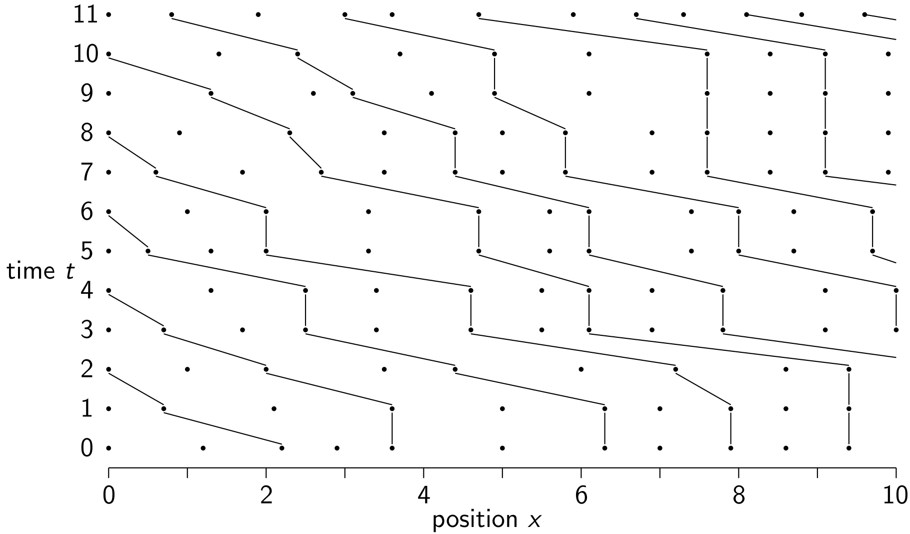
This phenomenon is easy to understand qualitatively. When a person leaves the checkpoint, the next person moves up to the checkpoint, the next person moves up and stops behind the now-first person, and so on, but this "wave" of motion often does not extend through the entire long line; instead, some person will move only a short distance, and the person behind will decide not to move at all.

Intuitively, when you are around the k'th position in line, there must be some number $a(k)$

- $a(k) =$ average time between your moves
- $a(k) =$ average distance you move when you do move
- $\mathbb{P}(W > k) = 1/a(k)$ for length of typical wave.

You are moving forwards at average speed 1 [unit time = service time, unit distance = average distance between people in queue]. This immediately suggests the question of how fast $a(k)$ grows with k.

I will present a stochastic model in which $a(k)$ grows as order $k^{1/2}$.



Space-time trajectories of alternate customers near the head of the queue.

In classical *queueing theory* randomness enters via assumed randomness of arrival and service times. In contrast, even though we are modeling a literal queue, randomness in our model arises in a quite different way, via each customer's choice of exactly how far behind the preceding customer they choose to stand, after each move. That is, we assume that "how far behind" is chosen (independently for each person and time) from a given probability density function μ on an interval $[c_-, c^+]$ where $c_- > 0$. We interpret this interval as a "comfort zone" for proximity to other people. By scaling we may assume μ has mean 1, and then (excluding the deterministic case) μ has some variance $0 < \sigma^2 < \infty$.
In words, the model is

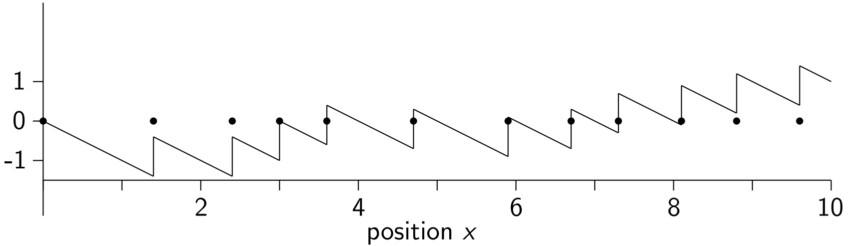
> *when the person in front of you moves forward to a new position, then you move to a new position at a random distance (chosen from distribution μ) behind them, unless their new position is less than distance c^+ in front of your existing position, in which case you don't move, and therefore nobody behind you moves.*

- Model **could** have been studied 60 years ago – but I can't find any closely related literature. Some traffic models loosely similar; also TASEP.
- Model as infinite queue.
- You might guess process has stationary distribution with inter-customer distances IID μ – **no**.
- Not obvious how to start analysis.
- Seem obvious that process time-converges to some unique stationary distribution – **cannot prove**.

It turns out there is a non-obvious picture which explains everything (intuitively).

A configuration $\mathbf{x} = (0 = x_0 < x_1 < x_2 < x_3 \ldots)$ of customer positions can be represented by its centered counting function

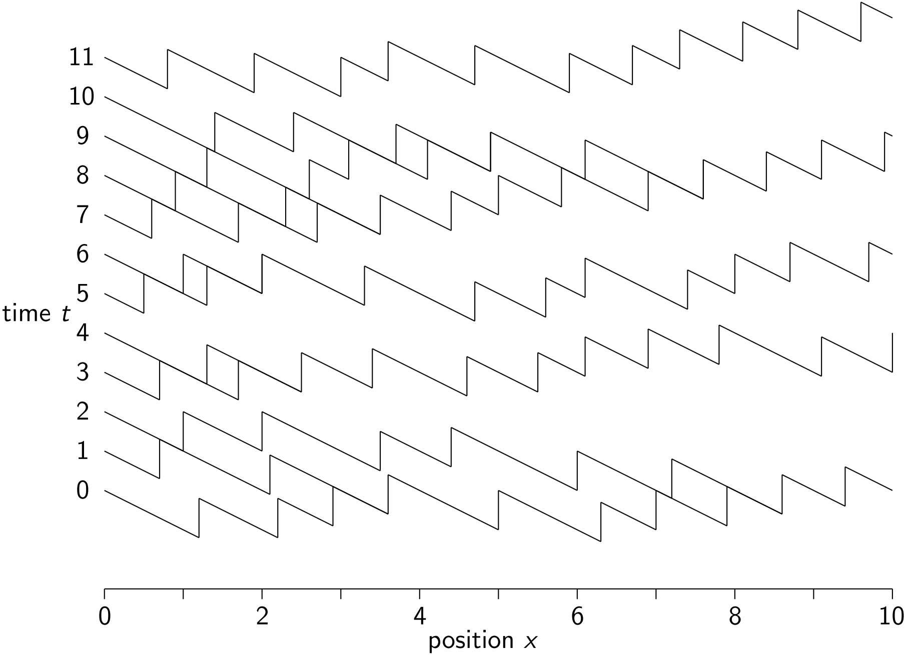
$$F(x) := \max\{k : \; x_k \le x\} \; - x, \quad 0 \le x < \infty. \tag{1}$$

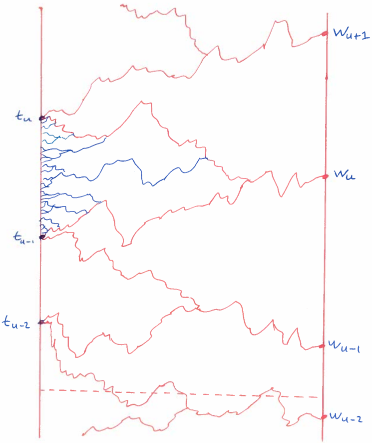


At each time t, let us consider the centered counting function $F_t(x)$ and plot the graph of the upward-translated function

$$x \to G(t,x) := t + F_t(x). \tag{2}$$

In other words, we draw the function starting at the point $(0, t)$ instead of the origin. Taking the same process realization as in the first Figure 1, superimposing all these graphs, gives the next Figure.

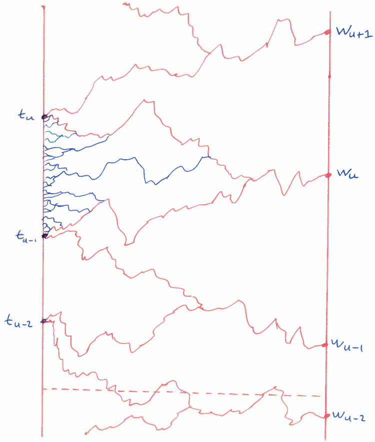

time t
0
1
2
3
4
5
6
7
8
9
10
11
0
2
4
6
8
10
position x



Picture shows **coalescing Brownian motion** (CBM).
Note trick: **we switched space** $\leftrightarrow$ **time**.

Assuming convergence of "coded" process to CBM, how do we decode?

- Rank and position same to first order – we are studying a second-order behavior.
- Space-time trajectories become vertical in the scaling limit.
- To study trajectory of individual at rank/position $\approx k$ there is a $k^{1/2}$ scaling in the vertical direction.



Trajectory of individual at rank/position $\approx k$ is

- Times between moves are $k^{1/2} \times$ intervals $t_{i+1} - t_i$ on left.
- Distances of moves are $k^{1/2} \times$ intervals $w_{i+1} - w_i$ on right.

How to actually prove the CBM limit?

AAAARGH !

30 pages with some details missing. Markov intuition fallible because of space $\leftrightarrow$ time switch. Must be some simpler proof ideas . . . . . .

Step -1.
Study W = length of wave at typical time. Suppose we can prove the desired order of magnitude

$$\mathbb{P}(W > w) \asymp w^{-1/2}.$$

Then we can lean on classical "random walk $\to_d$ BM" weak convergence, together with "robustness" of CBM – initial configuration unimportant for long-term behavior.

Step -2.
Study configuration at large time T by looking backwards. Can split customers into "blocks"; customers with a block all last moved at the same time [next slide].

[delicate] can couple time-T inter-customer distances $(\xi_i, i \geq 1)$ with IID(μ) $(\xi_i^*, i \geq 1)$ such that

$$\xi_i = \xi_i^* \text{ outside event } A_i$$

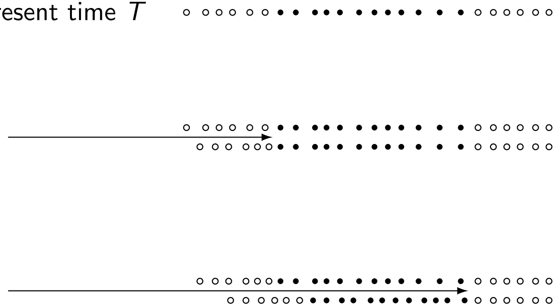
where A_i is a "block boundary" event.

Given the **upper** bound $\mathbb{P}(W > w) = O(w^{-1/2})$ we have $\mathbb{P}(A_i) = O(i^{-1/2})$ and then from the coupling

$$(*) \text{ rank-}k \text{ customer is at position } k \pm O(k^{1/2}).$$

Because rank decreases by 1 each time, this shows the customer must move during a typical $O(k^{1/2})$ time interval, and this is the **lower** bound $\mathbb{P}(W > w) \neq o(w^{-1/2})$.

present time T



Step -3. (First step). Want to prove **upper** bound

$$\mathbb{P}(W > w) = O(w^{-1/2}).$$

Given the current configuration with inter-customer distances $(\xi_i, i \geq 1)$, the event that next-step wave has length $> w$ is essentially the event

$$\{\sum_{i=1}^{n} \xi_i^* < \sum_{i=1}^{n} \xi_i \ \forall n < w\}$$

for IID(μ) (ξ_i^*). So **if** the current $(\xi_i, i \geq 1)$ were themselves IID(μ) then by standard RW excursion theory the probability would be $\asymp w^{-1/2}$. We now exploit the fact that, immediately after a long wave, the inter-customer distances are stochastically smaller than IID; this means that the time until the next long wave will be stochastically larger than if they were IID.

[end outline proof]

A more sophisticated approach might be to consider **CBM as a space-indexed Markov process.**

The standard CBM process $(B_y(s), 0 \le s < \infty, y \in \mathbb{R})$ with $B_y(0) = y$, with "time s" and "space" y, can in fact be viewed in the opposite way. Define

$$X_y = (B_y(s) - y,\ 0 \le s < \infty)$$

so that X_y takes values in the space $C_0(\mathbb{R}^+)$ of continuous functions f with $f(0) = 0$. Now the process $(X_y, -\infty < y < \infty)$ with "time" y is a continuous-time $C_0(\mathbb{R}^+)$-valued Markov process. Apparently CBM has not been studied explicitly in this way. In principle one could determine its generator and seek to apply general techniques for weak convergence of discrete-time Markov chains to continuous-time limits.

Not yet tried . . . . . .