Short-length routes in low-cost networks: expectation and variation

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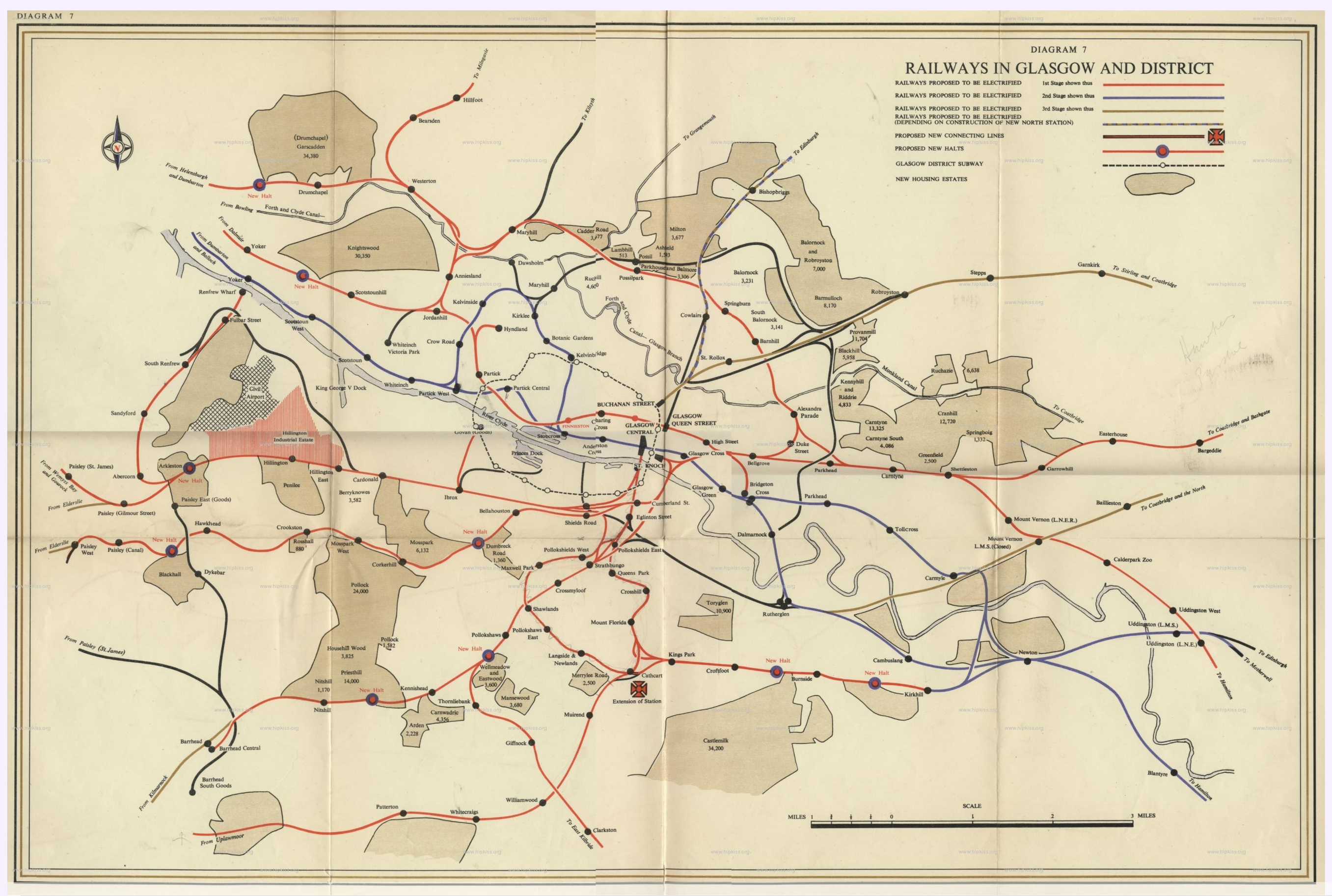
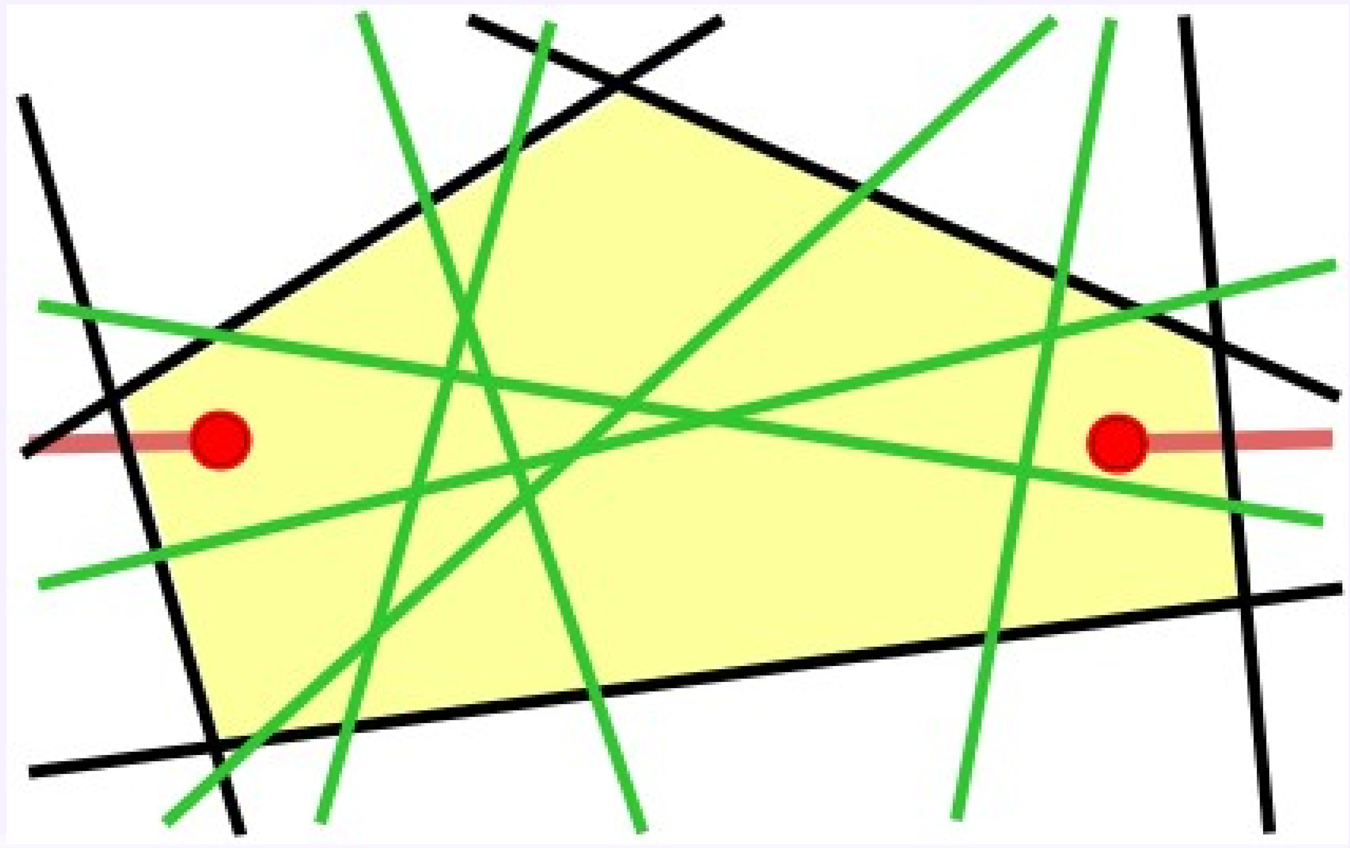


1: Rail networks, motorway networks, systems of Roman roads, ... all express **trade-offs** between (a) total cost (for example, in terms of total length) and (b) short routes (for example, in terms of average route length between two random nodes).



2: Associated question: what is total distance between two points as measured along network formed by Poisson line process? (Add two opposing rays to get from points onto network.)
Upper bound on network geodesic length: semi-perimeter of cell.

Theorem

$$\mathbb{E}\left[\operatorname{len}\partial\mathcal{C}_{x,y}\right] - 2|x-y| \quad = \quad \frac{\pi}{2} \times \iint_{\mathbb{R}^2} \frac{\alpha - \sin\alpha}{\pi} \exp\left(-\tfrac{1}{2}(\eta - n)\right) \mathrm{d}\,z$$

Theorem

Careful asymptotics for $n \to \infty$ show that

$$\mathbb{E}\left[\tfrac{1}{2}\operatorname{len}\partial\mathcal{C}_{x,y}\right] \quad = \quad n + \tfrac{1}{4}\iint_{\mathbb{R}^2}(\alpha - \sin\alpha)\exp\left(-\tfrac{1}{2}(\eta - n)\right)\mathrm{d}\,z \quad \approx \quad n + \frac{4}{3}\left(\log n + \gamma + \frac{5}{3}\right)$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

3: Use stochastic geometric techniques from stereology and duality to establish an upper bound, indicating a ***logarithmic* excess**.
The background illustrates extensive simulations which confirm these calculations. *Adv. Appl. Prob. 2008;* 40(1) 1-21.

Measure angle $\Theta(s)$ as function of accumulated excess arc-length s. Changes of direction arise according to Poisson($\frac{1}{2}$) process. New direction Θ_{n+1} determined in terms of old direction Θ_n *via*

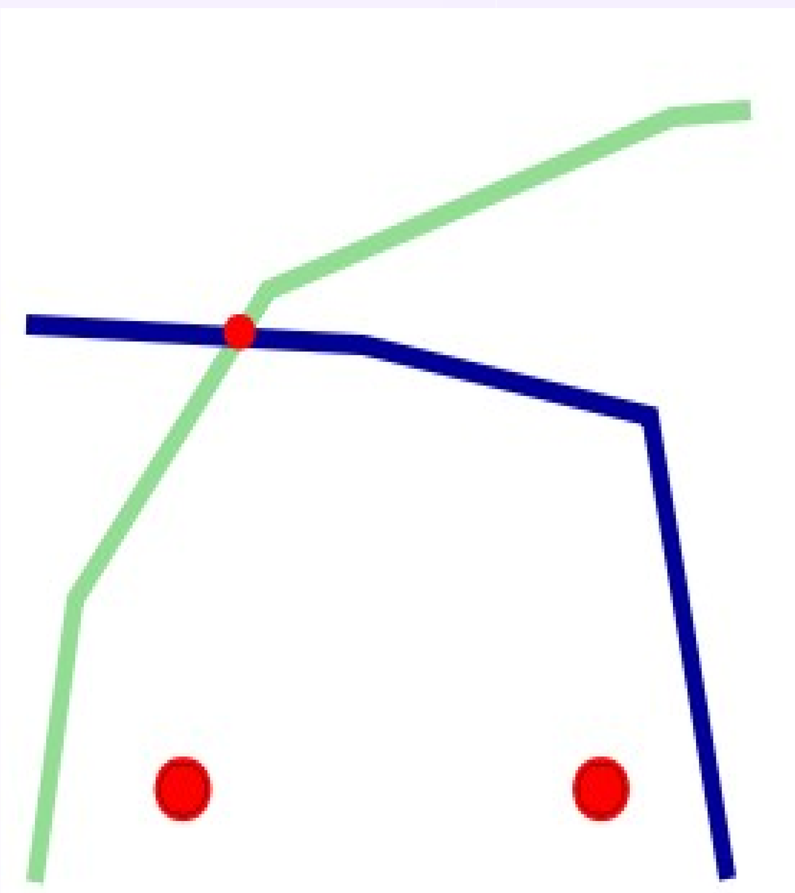
$$\mathbb{P}\left[\Theta_{n+1} \leq \phi | \Theta_n\right] \quad = \quad \frac{\cos(\Theta_n - \phi) - \cos(\Theta_n)}{1 - \cos(\Theta_n)}\,.$$

Actual distance travelled between changes of direction is

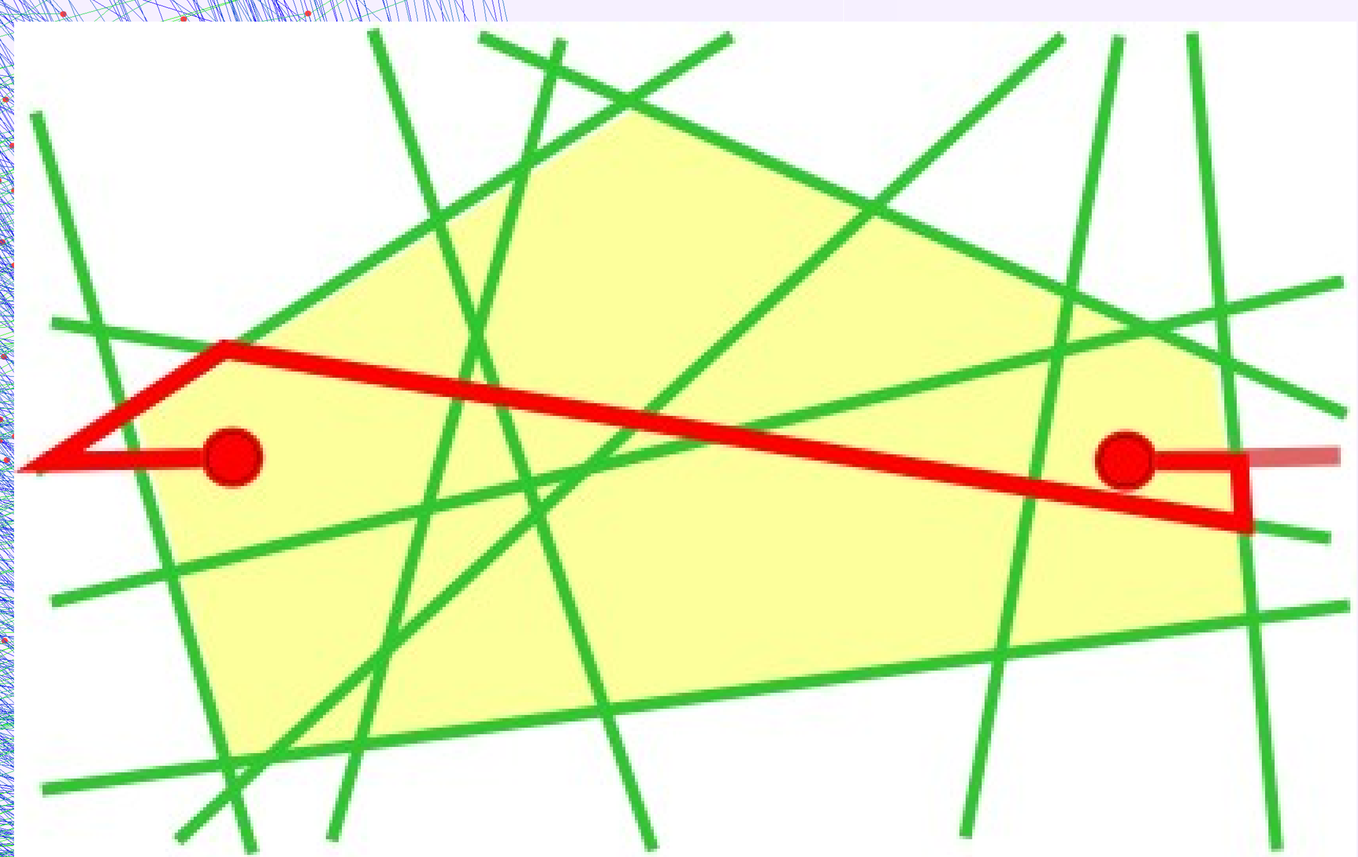
$$T_n \quad = \quad \frac{S_n}{1 - \cos(\Theta_n)}$$

where $S_n \sim$ Exponential($\frac{1}{2}$) is increase in accumulated excess.

4: What about **random variation**? Interpret the cell boundary in terms of *growth processes*. The green trajectories on the background illustrate growth processes from left; the blue trajectories illustrate growth from the right.



5: Asymptotics on these processes (a) confirm the stochastic geometry; (b) give information on the location of the maximum x (asymptotically uniform) and (c) give the conditional height of the maximum (asymptotically the length of a Gaussian 4-vector, Gaussian variance $n^2(1-4x^2/n^2)$).



6. This allows us to show that

(i) the **random variation of semi-perimeter length is order a constant**,

(ii) "short-cutting" across cell still results in logarithmic asymptotic excess.